APPLYING GENERIC CODING WITH HELP TO UNIFORMIZATIONS

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Abstract. This is a follow up to a paper by the author where the disjointness relation for (the graphs of) definable functions from ${}^{\omega}\omega$ to ${}^{\omega}\omega$ is analyzed. In that paper, for each $a \in {}^{\omega}\omega$ we defined a Baire class one function $f_a^{GC} : {}^{\omega}\omega \to {}^{\omega}\omega$ which encoded a in a certain sense. Given $g : {}^{\omega}\omega \to {}^{\omega}\omega$, let $\Psi(g)$ be the statement that g is disjoint from at most countably many of the functions f_a^{GC}. We show the consistency strength of $(\forall g)\, \Psi(g)$ is at most one inaccessible cardinal. We show that AD^+ implies $(\forall g)\, \Psi(g)$. Finally, we show that assuming large cardinals, $(\forall g)\, \Psi(g)$ holds in models of the form $L(\mathbb{R})[\mathcal{U}]$ where $\mathcal{U}$ is a selective ultrafilter on ω.

1. Introduction

We do not assume the Axiom of Choice in this paper unless explicitly stated. Our base theory is ZF. In [7] we isolated a lemma about Tree-Hechler Forcing. We review this as our Lemma 5.5 (the so called Main Lemma). One immediate consequence of this lemma, which is the focus of [6], is the following:

Theorem 1.1 (Generic Coding with Help). *If M is a countable transitive model of ZF and $x, y \in \mathbb{R}$ are reals such that $y \notin M$, then there is some Tree-Hechler generic G over M such that $x \in L[y, G]$.*

The proof of the Generic Coding with Help Theorem has many interesting consequences (which are explored in [6]), such as the following:

Corollary 1.2. *Let M be any transitive model of ZF. Let $\bar{a}$ be a set of ordinals not in M but such that $sup(\bar{a}) \in M$. Then there is a G that is set generic over M (which exists in a class forcing extension of V) such that $V \subseteq L[G][\bar{a}]$.*

However in this paper we take a step back and consider how to apply the Main Lemma to one area of descriptive set theory. Specifically,

A portion of the results of this paper were proven during the September 2012 Fields Institute Workshop on Forcing while the author was supported by the Fields Institute. Work was also done while under NSF grant DMS-0943832.

we apply it to uniformizations. Recall that given a binary relation $R \subseteq {}^\omega\omega \times {}^\omega\omega$ such that

$$(\forall x \in {}^\omega\omega)(\exists y \in {}^\omega\omega)\,(x,y) \in R,$$

we call $g : {}^\omega\omega \to {}^\omega\omega$ a *uniformization* of R iff

$$(\forall x \in {}^\omega\omega)\,(x, g(x)) \in R.$$

We will consider an edge case of this problem, where R is the complement of the graph of a function from ${}^\omega\omega$ to ${}^\omega\omega$. In other words, we have a function $f : {}^\omega\omega \to {}^\omega\omega$ and the problem is to find a function $g : {}^\omega\omega \to {}^\omega\omega$ such that

$$f \cap g = \emptyset$$

(the graphs of f and g are disjoint).

We show that there are several definable and uniform ways to map every real $a \in {}^\omega\omega$ to a function $f_a : {}^\omega\omega \to {}^\omega\omega$ such that if $g : {}^\omega\omega \to {}^\omega\omega$ is a "definable" function such that $f_a \cap g = \emptyset$, then a is in a countable and canonical set of reals associated to g (or rather, associated to an ∞-Borel code for g). We discuss two such mappings: one is $a \mapsto f_a^{PSP}$ which was explained to us by an anonymous referee, and a mapping $a \mapsto f_a^{GC}$ which was developed in [7] but restricted to projective functions g there.

First, let us describe a very coarse version of the problem:

Definition 1.3. A family of functions $\{f_a : a \in {}^\omega\omega\}$ from ${}^\omega\omega$ to ${}^\omega\omega$ indexed by ${}^\omega\omega$ such that $(a,x) \mapsto f_a(x)$ is Borel is called a *Borel family.*

Definition 1.4. Given functions $f, g : {}^\omega\omega \to {}^\omega\omega$, we say that g *avoids* f iff $f \cap g = \emptyset$.

Definition 1.5. Fix a family $\mathcal{F} = \{f_a : a \in {}^\omega\omega\}$ of functions from ${}^\omega\omega$ to ${}^\omega\omega$ and a function $g : {}^\omega\omega \to {}^\omega\omega$. We say that g *cannot avoid* $\mathcal{F}$ iff $\{a : f_a \cap g = \emptyset\}$ is countable. That is, g cannot avoid $\mathcal{F}$ iff g can avoid only countably many functions in $\mathcal{F}$.

Definition 1.6. Let Ψ be the statement that there is a Borel family of functions from ${}^\omega\omega$ to ${}^\omega\omega$ that no function from ${}^\omega\omega$ to ${}^\omega\omega$ can avoid.

Question 1.7. In what models does Ψ hold, and what families witness that Ψ holds?

Recall that the perfect set property (PSP) says that every uncountable set of reals contains a perfect subset. We will review this in Section 3. The PSP holds in both the Solovay model and in any model of AD. We show that the PSP implies that the Borel family

$\{f_a^{PSP} : a \in {}^\omega\omega\}$ cannot be avoided by any function from ${}^\omega\omega$ to ${}^\omega\omega$. Hence the PSP implies Ψ.

So now we know that $\{f_a^{PSP} : a \in {}^\omega\omega\}$ cannot be avoided in the Solovay model or in any model of AD. Similarly, we show by the Main Lemma about Tree-Hechler forcing that in the Solovay model or in any model of AD^+, $\{f_a^{GC} : a \in {}^\omega\omega\}$ cannot be avoided by any function from ${}^\omega\omega$ to ${}^\omega\omega$.

On the other hand, we show in Corollary 2.3 that ZFC implies $\neg\Psi$. We analyze that proof and conjecture in Section 4 that $\mathrm{DC}+\Psi$ implies $(\forall r \in {}^\omega\omega)\ \omega_1$ is inaccessible in $L[r]$.

Thus, here is our conjecture regarding consistency strengths:

Conjecture 1.8. *The following theories are equiconsistent.*

1) *ZFC* $+ \exists$ *an inaccessible cardinal;*
2) *ZF* $+ \mathrm{DC} + \mathrm{PSP}$*;*
3) *ZF* $+ \mathrm{DC} + \{f_a^{PSP} : a \in {}^\omega\omega\}$ *cannot be avoided;*
4) *ZF* $+ \mathrm{DC} + \{f_a^{GC} : a \in {}^\omega\omega\}$ *cannot be avoided;*
5) *ZF* $+ \mathrm{DC} + \Psi$*;*
6) *ZF* $+ \mathrm{DC} + \Psi$ *for only projective g's.*

The equiconsistency of 1) and 2) is well-known: one direction uses the Solovay model. We show in Section 3 that PSP implies $\{f_a^{PSP} : a \in {}^\omega\omega\}$ cannot be avoided, so 2) actually implies 3). The consistency of 1) implies the consistency of 4) using the Solovay model as we show in Section 7. Each of 3) and 4) imply 5) trivially, and 5) trivially implies 6). Finally, that 6) implies there is an inner model with an inaccessible cardinal is still a conjecture.

AD^+ is an axiom which implies AD, the Axiom of Determinacy, and it is open whether AD implies AD^+. The axiom AD^+ implies that every set of reals (and hence every function from ${}^\omega\omega$ to ${}^\omega\omega$) has a so called ∞-Borel code $C \subseteq \mathrm{Ord}$. We will define ∞-Borel codes soon.

We are also interested in the following question: given a Borel family such as $\{f_a : a \in {}^\omega\omega\}$ and a "definable" function g, what is a canonical description of a countable set $S \subseteq {}^\omega\omega$ such that S contains $\{a \in {}^\omega\omega : f_a \cap g = \emptyset\}$? For the family $\{f_a^{PSP} : a \in {}^\omega\omega\}$, we have the following:

Theorem 1.9. *Assume AD^+. Let $g : {}^\omega\omega \to {}^\omega\omega$ be a function. Let $Y \subseteq Ord$ be an ∞-Borel code for g. Then for any $a \in {}^\omega\omega$,*

$$[f_a^{PSP} \cap g = \emptyset] \to a \in \mathrm{HOD}_{\{Y\}}.$$

For the $\{f_a^{GC} : a \in {}^\omega\omega\}$ family, we prove a sharper result using the Main Lemma:

Theorem 1.10. *Assume AD^+. Let $g : {}^\omega\omega \to {}^\omega\omega$ be a function. Let $Y \subseteq Ord$ be an ∞-Borel code for g. Then for any $a \in {}^\omega\omega$,*

$$[f_a^{GC} \cap g = \emptyset] \to a \in L[Y].$$

Comparing the past two theorems, our intuition is that $\{f_a^{GC} : a \in {}^\omega\omega\}$ is "harder to avoid" than $\{f_a^{PSP} : a \in {}^\omega\omega\}$. We compare versions of the past two theorems but for projective functions $g : {}^\omega\omega \to {}^\omega\omega$ in Section 9.

Finally, this work suggests defining the following two regularity properties: $g : {}^\omega\omega \to {}^\omega\omega$ is PSP-regular iff it cannot avoid $\{f_a^{PSP} : a \in {}^\omega\omega\}$, and it is GC-regular iff it cannot avoid $\{f_a^{GC} : a \in {}^\omega\omega\}$. Projective Determinacy (PD) implies that every projective $g : {}^\omega\omega \to {}^\omega\omega$ is both PSP-regular and GC-regular (by Section 9). By Theorem 1.9 and Theorem 1.10 above, if $g : {}^\omega\omega \to {}^\omega\omega$ is in an inner model of AD^+ containing all the reals, then it is both PSP-regular and GC-regular. However, there may be more PSP-regular and GC-regular functions. That is, suppose there is a proper class of Woodin cardinals and CH holds. Let $\mathcal{U}$ be a selective ultrafilter on ω. Now $L(\mathbb{R})[\mathcal{U}]$ is a generic extension of $L(\mathbb{R})$ (see [10] and [4]). Using an argument pointed out to us by Paul Larson, the model $L(\mathbb{R})[\mathcal{U}]$ also satisfies the PSP. Thus, every $g : {}^\omega\omega \to {}^\omega\omega$ in $L(\mathbb{R})[\mathcal{U}]$ is PSP-regular. Our last result is Theorem 10.6 that states that in this same model $L(\mathbb{R})[\mathcal{U}]$, every g is GC-regular as well.

Remark 1.11. There is a different type of information that the disjointness relation can capture. Namely, assume AD. Fix $\alpha < \Theta$, where Θ is the smallest ordinal that ${}^\omega\omega$ cannot be surjected onto. Then there is a function $f : {}^\omega\omega \to {}^\omega\omega$ such that if $g : {}^\omega\omega \to {}^\omega\omega$ is any function that satisfies $g \cap f = \emptyset$, then g has Wadge rank $> \alpha$. We can construct f by diagonalizing over all functions of Wadge rank $\leq \alpha$: let $\langle h_x : x \in {}^\omega\omega \rangle$ be a logically simple enumeration of all continuous functions from ${}^\omega\omega \times {}^\omega\omega$ to ${}^\omega\omega$. Let $W \subseteq {}^\omega\omega$ be a set of Wadge rank α. For each $x \in {}^\omega\omega$, if $h_x^{-1}(W)$ is a function, define $f(x) := h_x^{-1}(W)(x)$. Otherwise, define $f(x)$ to be anything. Every Wadge rank $\leq \alpha$ function from ${}^\omega\omega$ to ${}^\omega\omega$ appears as some $h_x^{-1}(W)$.

Since ${}^\omega\omega \cong {}^\omega\omega \sqcup {}^\omega\omega$, we may combine this remark with Theorem 8.2 which we will prove. That is, assume AD^+. For every $\alpha < \Theta$ and for every $a \in {}^\omega\omega$, there is a function $f_{\alpha,a} : {}^\omega\omega \to {}^\omega\omega$ such that whenever $g : {}^\omega\omega \to {}^\omega\omega$ satisfies $f_{\alpha,a} \cap g = \emptyset$, then

1) g has Wadge rank $> \alpha$, and
2) $a \in L[Y]$ for any ∞-Borel code $Y \subseteq \text{Ord}$ for g.

1.1. **∞-Borel sets of reals.** Here is a concept we will use several times, so let us introduce it now:

Definition 1.12. A set $X \subseteq {}^\omega\omega$ is ∞-Borel iff there is a pair (C, φ), called an ∞-Borel code, such that C is a set of ordinals and φ is a formula such that

$$X = \{x \in {}^\omega\omega : L[C, x] \models \varphi(C, x)\}.$$

A similar definition applies to relations $R \subseteq {}^\omega\omega \times ... \times {}^\omega\omega$. We abuse language and call a set $C \subseteq \mathrm{Ord}$ an ∞-Borel code for $X \subseteq {}^\omega\omega$ iff there is a formula φ such that (C, φ) is an ∞-Borel code for X.

See [11] for more on ∞-Borel sets and AD^+ in general.

We do *not* define a function $g : {}^\omega\omega \to {}^\omega\omega$ to be ∞-Borel iff its graph is ∞-Borel: if C is an ∞-Borel code for the graph of $g : {}^\omega\omega \to {}^\omega\omega$, there is no guarantee that $g(x) \in L[C, x]$. This is the reason for the following definition:

Definition 1.13. A function $g : {}^\omega\omega \to {}^\omega\omega$ is ∞-Borel iff there is a pair (C, φ), called an ∞-Borel code, such that for all $x \in {}^\omega\omega$ and $n, m \in \omega$,

$$g(x)(n) = m :\Leftrightarrow L[C, x] \models \varphi(C, x, n, m).$$

We abuse language and call $C \subseteq \mathrm{Ord}$ an ∞-Borel code for $g : {}^\omega\omega \to {}^\omega\omega$ iff there is a formula φ such that (C, φ) is an ∞-Borel code for g.

We similarly define ∞-Borel codes for functions $g : {}^\omega\omega \to {}^\omega\omega \times [\omega]^\omega$, etc. We will sometimes be loose and write a code (C, φ) for the graph of g, but we will always mean the more technical definition. Note that if $g : {}^\omega\omega \to {}^\omega\omega$ is ∞-Borel with code C, then $g(x) \in L[C, x]$ for all x. Our strong definition of a function being ∞-Borel is justified because if every $A \subseteq {}^\omega\omega$ is ∞-Borel, then every $g : {}^\omega\omega \to {}^\omega\omega$ is ∞-Borel.

Example 1.14. Consider the function $g : {}^\omega\omega \to {}^\omega\omega$ defined by $g(x) = x'$, where x' is the Turing jump of x. This function is ∞-Borel with ∞-Borel code the empty set (because x' is definable in $L[x]$).

The following is important for us:

Fact 1.15. *Assume AD^+. Let X be a countable set of reals and let Y be an ∞-Borel code for X. Then $X \subseteq \mathrm{HOD}_{\{Y\}}$.*

Proof. See Fact 3.3 of [3]. □

2. Ψ is inconsistent with ZFC

Recall that Uniformization is the fragment of the Axiom of Choice that states that given any $R \subseteq {}^\omega\omega \times {}^\omega\omega$ satisfying $(\forall x \in {}^\omega\omega)(\exists y \in$

${}^{\omega}\omega)\,(x,y) \in R$, then there is a function $u : {}^{\omega}\omega \to {}^{\omega}\omega$ such that $u \subseteq R$. We call u a *uniformization* for R, or say that R is *uniformized* by u. Within this paper, we will not assume the Axiom of Choice unless explicitly stated. ZF is our base theory.

Proposition 2.1. *Uniformization* $+\,\Psi$ *implies that if* $S \subseteq {}^{\omega}\omega$ *is uncountable, then it can be surjected onto* ${}^{\omega}\omega$ *by a Borel function.*

Proof. Because we are assuming Ψ, fix a Borel family $\{f_a : a \in {}^{\omega}\omega\}$ that no function can avoid. Fix an uncountable set $S \subseteq {}^{\omega}\omega$. For each $x \in {}^{\omega}\omega$, the function $a \mapsto f_a(x)$ is Borel. We claim that for some $x \in {}^{\omega}\omega$, the function $a \mapsto f_a(x)$ surjects S onto ${}^{\omega}\omega$. Suppose this is not the case. For each $x \in {}^{\omega}\omega$, the set $Y_x := {}^{\omega}\omega - \{f_a(x) : a \in S\}$ is non-empty. Apply Uniformization to get $g : {}^{\omega}\omega \to {}^{\omega}\omega$ such that $(\forall x \in {}^{\omega}\omega)\, g(x) \in Y_x$. Then g is disjoint from f_a for each $a \in S$. Since S is uncountable, g avoids the family of f_a functions, which is a contradiction. $\square$

In Section 3 we will recall that if an uncountable set $S \subseteq {}^{\omega}\omega$ has a perfect subset, then S can be surjected onto ${}^{\omega}\omega$ by a Borel function. This is another indication that Ψ may be related to PSP.

It is clear that Ψ is inconsistent with ZFC $+\,\neg$CH, because given any $S \subseteq {}^{\omega}\omega$ of size $\omega_1 < 2^{\omega}$, there is a g disjoint from f_a for each $a \in S$. We will now show that Ψ is inconsistent with ZFC + CH as well. By Proposition 2.1, every uncountable $S \subseteq {}^{\omega}\omega$ can be surjected onto ${}^{\omega}\omega$ by a Borel function. Recall that $\text{add}(\mathcal{B})$ is the smallest size of a collection of meager sets of reals whose union is not meager (see [2] for more on $\text{add}(\mathcal{B})$, where it is called $\text{add}(\mathcal{M})$). We have $\omega_1 \le \text{add}(\mathcal{B}) \le 2^{\omega}$. This next proposition gives us our contradiction. Paul Larson pointed out how to make the diagonalization not get stuck by using the meager ideal.

Proposition 2.2. *Assume ZFC +* $\mathit{add}(\mathcal{B}) = 2^{\omega}$. *Then there exists a size* 2^{ω} *set* $S \subseteq {}^{\omega}\omega$ *that cannot be surjected onto* ${}^{\omega}\omega$ *by any Borel function.*

Proof. Because $\text{add}(\mathcal{B}) = 2^{\omega}$, the union of $< 2^{\omega}$ meager sets of reals is meager. For each Borel function h and each $y \in {}^{\omega}\omega$, $h^{-1}(y)$ has the property of Baire, so it is either comeager below a basic open set or it is meager. There can be only countably many y such that $h^{-1}(y)$ is comeager below some basic open set, because otherwise there would be two that intersect.

We now begin the construction of $S = \{a_\alpha : \alpha < 2^{\omega}\}$. Let $\langle h_\alpha : \alpha < 2^{\omega}\rangle$ be an enumeration of all Borel functions from ${}^{\omega}\omega$ to ${}^{\omega}\omega$. First, pick any $y_0 \in {}^\omega\omega$ such that $X_0 := h_0^{-1}(y_0)$ is meager. This y_0 will witness that h_0 does not surject S onto ${}^\omega\omega$. Now pick any $a_0 \in {}^\omega\omega - X_0$.

At stage $\alpha < 2^\omega$, pick any $y_\alpha \in {}^\omega\omega$ such that $X_\alpha := h_\alpha^{-1}(y_\alpha)$ is meager and does not contain any a_β for $\beta < \alpha$. This is possible because there are only $< 2^\omega$ many y such that $h_\alpha^{-1}(y)$ contains some a_β for $\beta < \alpha$, and there are only ω many y such that $h_\alpha^{-1}(y)$ is not meager. Then pick $a_\alpha \in {}^\omega\omega - \{a_\beta : \beta < \alpha\} - \bigcup_{\beta\leq\alpha} X_\beta$. When the construction finishes, the set S will have size 2^ω and for each $\alpha < 2^\omega$, $y_\alpha \notin h_\alpha(S)$. □

Corollary 2.3. *ZFC implies* $\neg\Psi$.

Proof. Assume, towards a contradiction, that ZFC + Ψ is consistent. Let us work within such a model. We previously gave a quick argument that ¬CH implies $\neg\Psi$, so it must be that CH holds. Thus $\mathrm{add}(\mathcal{B}) = 2^\omega$ holds. We also have Uniformization (because of the Axiom of Choice) and Ψ. Thus the hypothesis of the previous two propositions are satisfied. However, the conclusions of these propositions contradict one another. □

Remark 2.4. Miller [14] has shown that in the iterated perfect set model, in which $\omega_1 = \mathrm{add}(\mathcal{B}) < \omega_2 = 2^\omega$, every size ω_2 set $S \subseteq {}^\omega\omega$ can be surjected onto ${}^\omega\omega$ by a *continuous* function. The iterated perfect set model is obtained by starting with a model of CH and then adding ω_2 many Sacks reals by a countable support iteration. This leads us to the following question:

Question 2.5. Is it consistent with ZFC that there is a Borel family $\{f_a : a \in {}^\omega\omega\}$ such that every $g : {}^\omega\omega \to {}^\omega\omega$ is disjoint from only $< 2^\omega$ of the f_a functions? In such a model we would need ¬CH.

3. PSP implies Ψ

An anonymous referee has pointed out a certain family

$$\{f_a^{PSP} : a \in {}^\omega\omega\}$$

which witnesses Ψ when we assume the PSP. We will describe that in this section.

Fix a computable bijection from ω to ${}^{<\omega}\omega$ so that we may talk about coding a perfect tree $T \subseteq {}^{<\omega}\omega$ by a real $x \in {}^\omega\omega$. The set of reals through a perfect tree can be surjected onto ${}^\omega\omega$ in a uniform way that is Borel. We make this precise in the following lemma:

Lemma 3.1. *There is a Borel function* $E : {}^\omega\omega \times {}^\omega\omega \to {}^\omega\omega$ *such that for each* $x \in {}^\omega\omega$, *if* x *codes a perfect tree* $T_x \subseteq {}^\omega\omega$, *then*

$$\{E(x, y) : y \in [T_x]\} = {}^\omega\omega.$$

Hence, there is an $\alpha < \omega_1$ *such that for every perfect set* $S \subseteq {}^\omega\omega$*, there is a* $\mathbf{\Sigma}^0_\alpha$ *function from* ${}^\omega\omega$ *to* ${}^\omega\omega$ *that surjects* S *onto* ${}^\omega\omega$*.*

Proof. Fix an $x \in {}^\omega\omega$ that codes a perfect tree T_x. For each $y \in [T_y]$, let $D_x(y) \in {}^\omega 2$ be the sequence of 0's and 1's such that for each $n < \omega$, $D_x(y)(n)$ specifies whether y goes through the leftmost child of the n-th splitting node of T_x along y or whether it goes through a different child. Now for a fixed y, $D_x(y)$ can be considered as a sequence of s_0 many zeros, followed by a one, followed by s_1 many zeros, followed by a one, etc. Define $E(x,y) = \langle s_0, s_1, ...\rangle$. One can verify that $E``(\{x\} \times [T_x]) = {}^\omega\omega$ and also that the function E is Borel. □

Remark 3.2. Let $\mathcal{X} = ({}^\omega\omega)^n$ for some n. Let Γ be a pointclass. Suppose there is a set $U \subseteq {}^\omega\omega \times \mathcal{X}$ such that for each $B \subseteq \mathcal{X}$ in Γ, there is a $b \in {}^\omega\omega$ such that $B = \{y : (b,y) \in U\}$. Suppose also that U is in Γ. Then we call U a *universal set* and we call b a code for B. For each $\alpha < \omega_1$, the pointclass Σ^0_α has a universal set, so we may talk about codes for Σ^0_α sets.

Definition 3.3. Fix $\alpha < \omega_1$ such that every perfect subset of ${}^\omega\omega$ can be surjected onto ${}^\omega\omega$ by a $\mathbf{\Sigma}^0_\alpha$ function (such an α exists by the previous lemma). For each $a \in {}^\omega\omega$, let $f^{PSP}_a : {}^\omega\omega \to {}^\omega\omega$ be the function such that given any $x \in {}^\omega\omega$, if x codes a perfect tree $T_x \subseteq {}^{<\omega}\omega$ together with a $\mathbf{\Sigma}^0_\alpha$ surjection $s_x : [T_x] \to {}^\omega\omega$, and $a \in [T_x]$, then

$$f^{PSP}_a(x) := s_x(a).$$

Otherwise, $f^{PSP}_a(x)$ is the zero sequence.

We will define the f^{GC}_a functions later. However, it will be useful at this time to introduce the following notation:

Definition 3.4. Given $g : {}^\omega\omega \to {}^\omega\omega$,

$$D^{PSP}_g = \{a \in {}^\omega\omega : f^{PSP}_a \cap g = \emptyset\}$$
$$D^{GC}_g = \{a \in {}^\omega\omega : f^{GC}_a \cap g = \emptyset\}.$$

The following will be used later:

Lemma 3.5. *Let* Γ *be a pointclass containing all the Borel sets that is also closed under recursive substitutions. Let* $g : {}^\omega\omega \to {}^\omega\omega$ *be in* Γ *in the sense that the ternary relation "*$g(x)(n) = m$*" is in* Γ*. Let* $\{f_a : {}^\omega\omega \to {}^\omega\omega\}$ *be a Borel family. Let* $D_g := \{a \in {}^\omega\omega : f_a \cap g = \emptyset\}$*. Then* D_g *is* $\forall^{{}^\omega\omega}\neg\Gamma$*.*

Proof. A real $a \in {}^\omega\omega$ is in D_g iff

$$(\forall x \in {}^\omega\omega)(\forall n \in \omega)(\forall m \in \omega)[g(x)(n) = m \to f_a(x)(n) \neq m].$$

□

Here is the connection between the PSP and the f_a^{PSP} functions:

Lemma 3.6. *Fix a function $g : {}^\omega\omega \to {}^\omega\omega$. Then D_g^{PSP} cannot contain a perfect subset.*

Proof. Towards a contradiction, fix a perfect tree T such that $[T] \subseteq D_g^{PSP}$. Let $x \in {}^\omega\omega$ be such that $T_x = T$ and $s_x : [T] \to {}^\omega\omega$ is a surjection. So by definition of the f_a^{PSP} functions, we have $\{f_a^{PSP}(x) : a \in [T]\} = {}^\omega\omega$. Thus, $\{f_a^{PSP}(x) : a \in D_g^{PSP}\} = {}^\omega\omega$. In particular $g(x)$ is in this set, so fix $a \in D_g^{PSP}$ such that $f_a^{PSP}(x) = g(x)$. Thus $f_a^{PSP} \cap g \neq \emptyset$, which contradicts a being in D_g^{PSP}. □

Corollary 3.7. *Assume the PSP. Then for each $g : {}^\omega\omega \to {}^\omega\omega$, D_g^{PSP} is countable. Hence, Ψ holds as witnessed by the family $\{f_a^{PSP} : a \in {}^\omega\omega\}$.*

Proof. Assume, towards a contradiction, that there is some fixed g such that D_g^{PSP} is uncountable. Then D_g^{PSP} has a perfect subset $[T]$. This contradicts the lemma above. □

So the PSP implies each D_g^{PSP} is countable, but unfortunately we have no proof that PSP implies each D_g^{GC} is countable. Instead, we have a proof that D_g^{GC} is countable if either 1) AD^+ holds (see Corollary 8.3) or 2) we are in the Solovay model (see Corollary 7.2).

So it might seem that the family of f_a^{PSP} functions is strictly better than the family of f_a^{GC} functions. However, we have the interesting phenomenon that in nearly all instances where we can prove D_g^{GC} to be countable, we have a better bound on D_g^{GC} than we do for D_g^{PSP}. We explore this more in Section 9.

Assuming AD^+ we will prove

$$D_g^{PSP} \subseteq \text{HOD}_{\{Y\}}$$

whenever $Y \subseteq \text{Ord}$ is an ∞-Borel code for g. On the other hand, still with AD^+, in another section we will prove

$$D_g^{GC} \subseteq L[Y]$$

whenever $Y \subseteq \text{Ord}$ is an ∞-Borel code for g. The rest of the section will focus on the former result.

Proposition 3.8 (ZF)**.** *Let Y be a set of ordinals. Let $A \subseteq {}^\omega\omega$ be $\text{OD}_{\{Y\}}$ in the model $L(Y, \mathbb{R})$, where this model satisfies AD^+. Then A has an ∞-Borel code $S \subseteq Ord$ in $\text{HOD}_{\{Y\}}$.*

Proof. This follows by Theorem 10.2.6 in [11]. □

We now have the following:

Theorem 3.9. *Assume AD^+. Let $g : {}^\omega\omega \to {}^\omega\omega$ be a function. Let $Y \subseteq Ord$ be an ∞-Borel code for g. Then for any $a \in {}^\omega\omega$,*

$$[f_a^{PSP} \cap g = \emptyset] \to a \in \mathrm{HOD}_{\{Y\}}.$$

Proof. D_g^{PSP} is $\mathrm{OD}_{\{Y\}}$ in $L(Y, \mathbb{R})$. So D_g^{PSP} has an ∞-Borel code $S \subseteq$ Ord in $\mathrm{HOD}_{\{Y\}}$ by Proposition 3.8. By the PSP, D_g^{PSP} is countable. Thus $D_g^{PSP} \subseteq \mathrm{HOD}_{\{S\}}$ by Fact 1.15. Since $S \in \mathrm{HOD}_{\{Y\}}$ we have $\mathrm{HOD}_{\{S\}} \subseteq \mathrm{HOD}_{\{Y\}}$. Thus $D_g^{PSP} \subseteq \mathrm{HOD}_{\{Y\}}$. This is what we wanted to show. □

4. Consistency Strength Lower Bound of ZF + Ψ

In Section 2 we gave an argument that ZFC implies $\neg\Psi$. That is, ZFC implies that every Borel family $\{f_a : a \in {}^\omega\omega\}$ of functions from ${}^\omega\omega$ to ${}^\omega\omega$ can be avoided by some function $g : {}^\omega\omega \to {}^\omega\omega$. Using that argument and being careful about the complexity of the objects being produced, we will show in this section that $V = L$ implies every Borel family $\{f_a : a \in {}^\omega\omega\}$ of functions from ${}^\omega\omega$ to ${}^\omega\omega$ can be avoided by some $\mathbf{\Delta}^1_2$ function $g : {}^\omega\omega \to {}^\omega\omega$. We will convert this into a conjecture that ZF + DC + "there exists a Borel family that cannot be avoided" implies that ω_1 is inaccessible in $L[r]$ for each $r \in {}^\omega\omega$.

Remark 4.1. Temporarily suppose Γ is a pointclass closed under quantification of natural numbers. Let $\Delta = \Gamma \cap \neg\Gamma$. Let $g : {}^\omega\omega \to {}^\omega\omega$. Consider the ternary relation "$g(x)(n) = m$". Since

$$g(x)(n) \neq m \Leftrightarrow (\exists i \in \omega)\, i \neq m \wedge g(x)(n) = i,$$

we have that the ternary relation is in Γ iff it is in the dual $\neg\Gamma$. Since

$$g(x) = y \Leftrightarrow (\forall n \in \omega)[(\forall m \in \omega)\, m = y(n) \to g(x)(n) = m],$$

if the ternary relation "$g(x)(n) = m$" is in Γ then the binary relation "$g(x) = y$" is in Γ. Similarly, since

$$g(x)(n) = m \Leftrightarrow (\exists y \in {}^\omega\omega)[g(x) = y \wedge y(n) = m],$$

$$g(x)(n) = m \Leftrightarrow (\forall y \in {}^\omega\omega)[g(x) = y \Rightarrow y(n) = m],$$

if the binary relation is in Γ, then the ternary relation is in $\exists^{{}^\omega\omega}\Gamma$ and $\forall^{{}^\omega\omega}\Gamma$. By what we said about the binary relation versus the ternary relation, we have that the following are equivalent: Now fix an $1 \leq n < \omega$.

1) The binary relation "$g(x) = y$" is Σ^1_n.
2) The binary relation is Π^1_n.
3) The binary relation is Δ^1_n.
4) the ternary relation "$g(x)(n) = m$" is Σ^1_n

5) the ternary relation is Π^1_n
6) the ternary relation is Δ^1_n.

Using a definition of [15], a well-ordering $\leq$ of ${}^\omega\omega$ is called Γ-good iff it is in Γ and whenever P is a binary Γ-relation, then the relations $Q(x,y) \Leftrightarrow (\exists x' \leq x)\, P(x',y)$ and $R(x,y) \Leftrightarrow (\forall x' \leq x)\, P(x',y)$ are in Γ. Note that if $\leq$ is Γ-good, then it is also $\neg\Gamma$-good. If $V = L[r]$ for some $r \in {}^\omega\omega$, then there is a $\Sigma^1_2(r)$-good well-ordering of ${}^\omega\omega$ of order type ω_1.

We will follow Remark 3.2 in the construction below. That is, for $\alpha < \omega_1$, we will use codes to talk about the c-th $\mathbf{\Sigma}^0_\alpha$ function $h_c : {}^\omega\omega \to {}^\omega\omega$, where $c \in {}^\omega\omega$. That is, we fix a universal $\mathbf{\Sigma}^0_\alpha$ set and use its sections to get all the $\mathbf{\Sigma}^0_\alpha$ functions from ${}^\omega\omega$ to ${}^\omega\omega$.

Lemma 4.2. *Let $\leq$ be a Γ-good well-ordering of ${}^\omega\omega$. Let $P \subseteq {}^\omega\omega \times {}^\omega\omega$ be a binary Δ-relation such that $(\forall y \in {}^\omega\omega)(\exists x \in {}^\omega\omega)\, P(x,y)$. Then the relation $P' \subseteq {}^\omega\omega \times {}^\omega\omega$, defined by $P'(x,y) :=$ x is the $\leq$-least real satisfying $P(x,y)$, is also a Δ-relation.*

Proof. We can assume that the relation $R(a,b) := (a = b)$ is Δ.

$$P'(x,y) = P(x,y) \wedge (\forall x' \leq x)[x' \neq x \to \neg P(x,y)]$$

is Γ, because $\neg P(x,y)$ is Γ (because P is Δ) and so $x' \neq x \to \neg P(x,y)$ is Γ and so on. On the other hand,

$$\neg P'(x,y) := \neg P(x,y) \vee (\exists x' \leq x)[x' \neq x \wedge P(x',y)]$$

is Γ, and so P' is $\neg\Gamma$. Thus P' is Δ. □

Definition 4.3. Fix a computable bijection from ω to $\omega \times \omega$. Given a relation $R \subseteq \omega \times \omega$, we may use that bijection to encode R as a subset of ω, which we can then identity as an element of ${}^\omega\omega$. In this way, given a hereditarily countable set S, call $c \in {}^\omega\omega$ an $H(\omega_1)$ *code* for S iff c codes a binary relation $R \subseteq \omega \times \omega$ that is isomorphic to the $\in$ relation on the transitive closure of $S \cup \{S\}$ such that if we let $\pi_c : \langle \omega, R \rangle \to \langle \mathrm{TC}(S \cup \{S\}), \in \rangle$ be the isomorphism, then $\pi_c(0) = S$.

Note that the set of all $H(\omega_1)$ codes is a Π^1_1 set. One point of $H(\omega_1)$ codes is to convert the quantification over the elements of a fixed hereditarily countable S to a number quantifier using a code for S as a parameter. Specifically, given an $H(\omega_1)$ code c for a hereditarily countable set S, the set $S \cap {}^\omega\omega$ of reals in S is $\Delta^1_1(c)$.

Theorem 4.4. *Fix $2 \leq n < \omega$. Assume that* CH *holds. Assume also there is a $\mathbf{\Sigma}^1_n$-good well-ordering $\leq$ of ${}^\omega\omega$ of order type ω_1. Fix $\alpha < \omega_1$. There is an uncountable set $S \subseteq {}^\omega\omega$ along with a $\mathbf{\Delta}^1_n$ function*

$H : {}^\omega\omega \to {}^\omega\omega$ such that whenever $c \in {}^\omega\omega$ is a code for a $\mathbf{\Sigma}^0_\alpha$ function $h : {}^\omega\omega \to {}^\omega\omega$, then $H(c) \notin h``S$. That is, H witnesses that no $\mathbf{\Sigma}^0_\alpha$ function surjects S onto ${}^\omega\omega$.

Proof. For each $x \in {}^\omega\omega$, by induction on the $\le$-rank of x, define a pair $(a_x, y_x) \in {}^\omega\omega \times {}^\omega\omega$ as follows. Also, note by induction that each pair (a_x, y_x) is unique.

1) y_x is the $\le$-least real such that
   1a) $h_x^{-1}(y_x)$ is meager;
   1b) $(\forall x' \le x)\, x' \ne x \Rightarrow a_{x'} \notin h_x^{-1}(y_x)$.
2) a_x is the $\le$-least real such that
   2a) $(\forall x' \le x)\, x' \ne x \Rightarrow a_x \ne a_{x'}$;
   2b) $(\forall x' \le x)\, a_x \notin h_{x'}^{-1}(y_{x'})$.

Before we proceed, let us prove that (a_x, y_x) exists. Fix x and suppose we have defined $(a_{x'}, y_{x'})$ for all $x' < x$. We will find the (a_x, y_x). First, we will define the y_x that works. Since there are only countably many y's such that $a_{x'} \in h_x^{-1}(y_x)$ (because $\{a_{x'} : x' < x\}$ is countable), we can just throw out those y's when trying to satisfy 1b). There are only countably many y's such that $h_x^{-1}(y)$ is not meager. This is because h_x is Borel so each $h_x^{-1}(y)$ is Borel. For each y such that $h_x^{-1}(y)$ is not meager, we may pick a non-empty open set U_y such that the symmetric difference between $h_x^{-1}(y)$ and U_y is meager. Now if there are uncountably many y's such that $h_x^{-1}(y)$ is non-meager, then there must be two of the corresponding U_y's whose intersection is non-empty (and open). A contradiction easily follows. Thus, we can throw away just countably many y's to satisfy 1a). To summarize, we threw away the countably many y's that did not satisfy 1a) and we threw away the countably many y's that did not satisfy 1b), so we are left with the cocountable set of y's that satisfy both 1a) and 1b). We pick y_x to be the least such y.

Now we must pick an a_x that satisfies 2). Getting a_x to satisfy 2a) is easy because $\{a_{x'} : x' < x\}$ is countable. That is, there is a cocountable (and hence comeager) set of a's that work for 2a). For 2b), since each $h_{x'}^{-1}(y_{x'})$ is meager, and the union of these for $x' \le x$ is also meager and hence its complement is comeager. Thus the set of a_x's that work for 2) is comeager, and so we can pick a_x to be the $\le$-least such one.

Let our set S be

$$S = \{a_x : x \in {}^\omega\omega\}.$$

Note that by 2a), S is uncountable. Let $H : {}^\omega\omega \to {}^\omega\omega$ be the function

$$H(x) := y_x.$$

We must now do two things: show that H witnesses that no $\mathbf{\Sigma}^0_\alpha$ function surjects S onto ${}^\omega\omega$, and show H is $\mathbf{\Delta}^1_n$. We will do the former first.

Note that by 1b) and 2b) together, we have

$$(\forall x_1, x_2 \in {}^\omega\omega)\, a_{x_1} \not\in h^{-1}_{x_2}(y_{x_2}).$$

That is,

$$(\forall x_1, x_2 \in {}^\omega\omega)\, h_{x_2}(a_{x_1}) \neq y_{x_2}.$$

Now fix a $\mathbf{\Sigma}^0_\alpha$ function $h : {}^\omega\omega \to {}^\omega\omega$. We will show that h does not surject S onto ${}^\omega\omega$. Fix an $x_2 \in {}^\omega\omega$ such that $h_{x_2} = h$. We now have

$$(\forall x_1 \in {}^\omega\omega)\, h_{x_2}(a_{x_1}) \neq y_{x_2}.$$

So y_{x_2} is not in the range of $h \restriction S$.

The last thing we must do is show H is $\mathbf{\Delta}^1_n$. For each $c \in {}^\omega\omega$, let F_c be the function $x \mapsto (a_x, y_x)$ restricted to $\{x : x \le c\}$. Since F_c is hereditarily countable, it has an $H(\omega_1)$ code. Let $J : {}^\omega\omega \to {}^\omega\omega$ be the function defined by $J(c) :=$ the $\le$-least $H(\omega_1)$ code for F_c. Consider the relation $R \subseteq {}^\omega\omega \times {}^\omega\omega$ defined by $R(d, c) :=$"d is an $H(\omega_1)$ code for F_c". We will show that R is $\mathbf{\Delta}^1_n$. It will follow that J is $\mathbf{\Delta}^1_n$ (by the proof of Lemma 4.2), and so H is $\mathbf{\Delta}^1_n$.

Note that the well-ordering $\le$ is in fact $\mathbf{\Delta}^1_n$. In this paragraph we will show that the relation $R_1 \subseteq {}^\omega\omega \times {}^\omega\omega$ defined by $R_1(d, c) :=$ "$d \in {}^\omega\omega$ is an $H(\omega_1)$ code for $\{x : x \le c\}$" is $\mathbf{\Delta}^1_n$. Quantifying over the reals in the countable set coded by a d is a number quantifier, not a real quantifier. So, "$(\forall x \in$ the set coded by $d)\, x \le c$" is $\mathbf{\Delta}^1_n$. On the other hand, "$(\forall x \le c)\, x \in$ the set coded by d" is $\mathbf{\Delta}^1_n$ because $\le$ is $\mathbf{\Sigma}^1_n$-good and $\mathbf{\Pi}^1_n$-good. This shows that the relation R_1 is $\mathbf{\Delta}^1_n$. Similarly $R_2(d, c) :=$ "$d \in {}^\omega\omega$ is an $H(\omega_1)$ code for a function from $\{x : x \le c\}$ to ${}^\omega\omega \times {}^\omega\omega$" is $\mathbf{\Delta}^1_n$.

We will now prove that "$R(d, c) := d$ is an $H(\omega_1)$ code for F_c is $\mathbf{\Delta}^1_n$", and this will complete the proof. Because $\le$ is $\mathbf{\Sigma}^1_n$-good and $\mathbf{\Pi}^1_n$-good, it suffices to show that 1) and 2) are $\mathbf{\Delta}^1_n$. First, consider 1a). Each set $h^{-1}_x(y_x)$ is Borel, and we can uniformly get a code for this set from x and y_x. Given $\beta < \omega_1$, whether or not a code for a $\mathbf{\Sigma}^0_\beta$ set codes a meager set is certainly $\mathbf{\Delta}^1_n$. Next, since "$a_{x'} \not\in h^{-1}_x(y_x)$" is $\mathbf{\Delta}^1_n$ and $\le$ is $\mathbf{\Sigma}^1_n$-good and $\mathbf{\Pi}^1_n$-good we have that 1b) is $\mathbf{\Delta}^1_n$. So, the conjunction of 1a) and 1b) is $\mathbf{\Delta}^1_n$. The property of being the $\le$-least real that satisfies a $\mathbf{\Delta}^1_n$ relation is $\mathbf{\Delta}^1_n$, so it follows that 1) is $\mathbf{\Delta}^1_n$.

Now "$a_x \neq a_{x'}$" is certainly $\mathbf{\Delta}^1_n$, so 2a) is $\mathbf{\Delta}^1_n$ because $\le$ is $\mathbf{\Sigma}^1_n$-good and $\mathbf{\Pi}^1_n$-good. Similarly, 2b) is $\mathbf{\Delta}^1_n$. Now the conjunction of 2a) and 2b) is $\mathbf{\Delta}^1_n$, and so 2) is $\mathbf{\Delta}^1_n$ as well. □

Corollary 4.5. *Fix $2 \leq n < \omega$. Assume* CH *holds and there is a $\mathbf{\Sigma}^1_n$-good well-ordering $\leq$ of ${}^\omega\omega$ of order type ω_1. Let $\{f_a : a \in {}^\omega\omega\}$ be a Borel family of functions from ${}^\omega\omega$ to ${}^\omega\omega$. Then there is a $\mathbf{\Delta}^1_n$ function $g : {}^\omega\omega \to {}^\omega\omega$ that avoids the family.*

Proof. Fix $\alpha < \omega_1$ such that each function $a \mapsto f_a(x)$ is $\mathbf{\Sigma}^0_\alpha$. Let S and H be from the Lemma above. Define $g : {}^\omega\omega \to {}^\omega\omega$ as follows. There is a Borel function $x \mapsto c_x$ such that for each $x \in {}^\omega\omega$, the real c_x is a code for the function $a \mapsto f_a(x)$. Fix such a function. Now for all $x \in {}^\omega\omega$, we have $H(c_x) \notin \{f_a(x) : a \in S\}$. Define $g(x) := H(c_x)$. We now have for each $a \in S$ that $f_a \cap g = \emptyset$. Thus since S is uncountable, g avoids the family $\{f_a : a \in {}^\omega\omega\}$. Also, one can check that g is in fact $\mathbf{\Delta}^1_n$. □

Remark 4.6. Here are some ways to apply the corollary above. First, $V = L$ implies there is a Σ^1_2-good well ordering of ${}^\omega\omega$ of order type ω_1. Going up the large cardinal hierarchy, if LC is a large cardinal axiom consistent with there being a Σ^1_n-good well-ordering of ${}^\omega\omega$ of order type ω_1 (for some fixed $n < \omega$), then in such a model we have that for each Borel family $\{f_a : a \in {}^\omega\omega\}$ of functions from ${}^\omega\omega$ to ${}^\omega\omega$, there is a $\mathbf{\Delta}^1_n$ function $g : {}^\omega\omega \to {}^\omega\omega$ that avoids the family. So for example, assuming there are only finitely many Woodin cardinals does not imply that every projective function avoids $\{f_a^{PSP} : a \in {}^\omega\omega\}$. See [19] for a discussion of the mouse $M_n^\#$ which has $n \in \omega$ Woodin cardinals but at the same time a Δ^1_{n+2}-good well-ordering of $\mathbb{R}$.

We would like to show that if there is a Borel family of functions that cannot be avoided by a projective function, then there is an inner model with an inaccessible cardinal. However our argument relies on the following conjecture:

Conjecture 4.7. *Let $\{f_a : a \in {}^\omega\omega\}$ be a Borel family of functions. Let $x \in \mathbb{R}$ be such that $\omega_1 = \omega_1^{L[x]}$. Let $S \subseteq {}^\omega\omega$ be a set of reals that is in $L[x]$ and is uncountable there. Then if in $L[x]$ there is a $\mathbf{\Delta}^1_2$ function g that is disjoint from $f_a \restriction L[x]$ for each $a \in S$, then there is a projective function $g^+ : {}^\omega\omega \to {}^\omega\omega$ that extends g (in V) that is disjoint from f_a for each $a \in S$.*

Theorem 4.8. *Assume* DC*. Assume Conjecture 4.7. Assume there is a Borel family $\{f_a : a \in {}^\omega\omega\}$ such that no projective function $g : {}^\omega\omega \to {}^\omega\omega$ can avoid this family. Then $(\forall r \in {}^\omega\omega)\, r$ is inaccessible in $L[r]$.*

Proof. Fix a Borel family $\{f_a : a \in {}^\omega\omega\}$. Fix $b \in {}^\omega\omega$ such that the function $(a, x) \mapsto f_a(x)$ is $\Delta^1_1(b)$. Since we are assuming $\mathrm{ZF} + \mathrm{DC}$, the statement $(\forall r \in {}^\omega\omega)\, \omega_1$ is inaccessible in $L[r]$ is equivalent to the statement $(\forall r \in {}^\omega\omega)\, \omega_1^{L[r]} < \omega_1$ [8]. We will prove the contrapositive. That is, fix $r \in {}^{\omega}\omega$ such that $\omega_1^{L[r]} = \omega_1$. So we also have $\omega_1^{L[r,b]} = \omega_1$. We will construct a projective function that is disjoint from uncountably many of the f_a functions (so the projective function avoids the family of f_a functions).

Note that in $L[r,b]$, there is a $\mathbf{\Sigma}^1_2$-good well-ordering of ${}^{\omega}\omega$ of order type ω_1. Apply Corollary 4.5 above in $L[r,b]$ to get $S \subseteq {}^{\omega}\omega \cap L[r,b]$ uncountable (in $L[r,b]$) and let $g \in L[r,b]$ be $\mathbf{\Delta}^1_2$ (in $L[r,b]$) such that g is disjoint (in $L[r,b]$) from each f_a for $a \in S$. By Conjecture 4.7 fix a projective function $g^+ : {}^{\omega}\omega \to {}^{\omega}\omega$ (in V) that is disjoint (in V) from f_a for each $a \in S$. Thus, the projective function g^+ avoids the family $\{f_a : a \in {}^{\omega}\omega\}$ which is what we wanted to show. □

Corollary 4.9. *Assume* DC. *Assume Conjecture 4.7. Then* Ψ *implies that* $(\forall r \in {}^{\omega}\omega)$ ω_1 *is inaccessible in* $L[r]$.

5. f_a^{GC} AND $\mathbb{H}$

In this section we will review the technology of the *Generic Coding with Help* method. A key ingredient is a classical technique for generating an infinite subset of ω that is computable from every infinite subset of itself (such a set is called *introreducible*). We review this first:

Proposition 5.1. *Let* $X \subseteq \omega$. *There is an infinite* $Y \subseteq \omega$ *such that* X *is computable from every infinite subset of* Y. *Moreover,* Y *can be taken to be Turing equivalent to* X.

Proof. Let $\chi : \omega \to 2$ be the characteristic function of X. Let $p_0, p_1, p_2, ...$ be the increasing enumeration of all the prime numbers. Let $Y \subseteq \omega$ be the set of all numbers of the form $p_0^{\chi(0)} p_1^{\chi(1)} ... p_n^{\chi(n)}$ for all $n \in \omega$. Then Y is as desired. This encoding trick is sometimes called "stuttering". Indeed, we can see that given any $m \in Y$, by finding the prime factorization of m we can read off an initial segment of χ. If we have infinitely many such m's, then we can recover all of χ. It is not hard to see that X and Y are Turing equivalent. □

Once and for all, fix a Borel function that maps each real $a \in {}^{\omega}\omega$ to an infinite set $A_a \subseteq \omega$ such that 1) a and A_a are Turing equivalent and 2) A_a is computable from every infinite subset of itself. Now for each $a \in {}^{\omega}\omega$, we will define the function $f_a^{GC} : {}^{\omega}\omega \to {}^{\omega}\omega$.

Definition 5.2. Fix a computable function $\theta : \omega \to \omega$ such that

$$(\forall m \in \omega)\, \theta^{-1}(m) \text{ is infinite.}$$

Given an $a \in {}^{\omega}\omega$, let $e_a : \omega \to A_a$ be the strictly increasing enumeration of A_a. Let $\eta_a : A_a \to \omega$ be the function $\theta \circ e_a^{-1}$.

Note that for each $m \in \omega$, $\eta_a^{-1}(m) \subseteq A_a$ is infinite.

Definition 5.3. The function $f_a^{GC} : {}^\omega\omega \to {}^\omega\omega$ is defined as follows: Given $x = \langle x_0, x_1, ...\rangle \in {}^\omega\omega$, let $i_0 < i_1 < ...$ be the indices i such that $x_i \in A_a$. Define f_a^{GC} to be

$$f_a^{GC}(x) := \langle \eta_a(x_{i_0}), \eta_a(x_{i_1}), ...\rangle.$$

If there are only finitely many indices i such that $x_i \in A_a$, then define $f_a^{GC}(x)$ to be all 0's after these finitely many indices.

Remark 5.4. Note that $(a, x) \mapsto f_a^{GC}(x)$ is Borel.

To see how the coding works, consider a node $t \in {}^{<\omega}\omega$. Let $n \in \omega$ be the number of $l \in \mathrm{Dom}(t)$ such that $t(l) \in A_a$. All $x \in {}^\omega\omega$ that extend t agree up to the first n values of $f_a(x)$, but not at the $(n+1)$-th value. By extending t by one to get $t^\frown k$ for some $k \in A_a$, we can decide the $(n+1)$-th value of $f_a(x)$ to be anything we want. Even if there is a finite set S of k which we are not allowed to pick, we can still create a $t^\frown k$ where the $(n+1)$-th value of $f_a(x)$ is anything we want.

The poset $\mathbb{H}$, a variant of Hechler forcing, is equivalent to the forcing which consists of trees $T \subseteq {}^{<\omega}\omega$ with co-finite splitting after the stem, where the ordering $\leq$ is reverse inclusion. We present $\mathbb{H}$ as consisting of pairs (t, h) such that $t \in {}^{<\omega}\omega$ and $h : {}^{<\omega}\omega \to \omega$, where t specifies the stem and h specifies where each node beyond the stem has a final segment of successors. That is, we have $(t', h') \leq (t, h)$ iff $h' \geq h$ (everywhere domination), $t' \sqsupseteq t$, and for each $n \in \mathrm{Dom}(t') - \mathrm{Dom}(t)$,

$$t'(n) \geq h(t' \restriction n).$$

Given a set $A \subseteq \omega$, there is also a stronger ordering $\leq^A$ defined by $(t', h') \leq^A (t, h)$ iff $(t', h') \leq (t, h)$ and for each $n \in \mathrm{Dom}(t') - \mathrm{Dom}(t)$,

$$t'(n) \notin A.$$

Informally, $q \leq^A p$ iff $q \leq p$ and the stem of q does not "hit" A any more than p already does. We will also use the main lemma from [7], which tells us a situation where we can hit a dense subset of $\mathbb{H}$ by making a $\leq^A$ extension. By an ω-model we mean a model of ZF that is possibly ill-founded but whose ω is well-founded (and so equal to the true ω). Moreover, this next lemma only needs M to satisfy a fragment of ZF.

Lemma 5.5. *(Main Lemma) Let M be an ω-model of ZF and $D \in \mathcal{P}^M(\mathbb{H}^M)$ a set dense^M in $\mathbb{H}^M$. Let $A \subseteq \omega$ be infinite and Δ^1_1 in every infinite subset of itself, but $A \notin M$. Then*

$$(\forall p \in \mathbb{H}^M)(\exists p' \leq^A p)\, p' \in D.$$

6. Abstract f_a^{GC} Theorem

The point of this next theorem is that if a model M of ZF can understand a function $g : {}^\omega\omega \to {}^\omega\omega$ on all its generic extensions by the $\mathbb{H}$ poset, and if $a \in {}^\omega\omega - M$, then we can build a real x that is $\mathbb{H}$-generic over M such that $f_a(x) = g(x)$. This is proved using the Generic Coding with Help method described in the previous section.

Theorem 6.1. *(ZF) Let M be a transitive model of ZF such that $\mathcal{P}^M(\mathbb{H}^M)$ is countable. Let $g : {}^\omega\omega \to {}^\omega\omega$. Let $\dot\tau$ be an $\mathbb{H}^M$-name for a function from ${}^\omega\omega \times \omega$ to ω such that for every $G \subseteq \mathbb{H}^M$ that is $\mathbb{H}^M$-generic over M, if we let $x = \bigcup\{t : (\exists h)\,(t,h) \in G\}$, then $(\forall n \in \omega)\, \dot\tau_G(x)(n) = g(x)(n)$. Then for all $a \in {}^\omega\omega$,*

$$[f_a \cap g = \emptyset] \Rightarrow a \in M.$$

Proof. Fix a and assume $a \notin M$. We must construct an $x \in {}^\omega\omega$ such that $f_a(x) = g(x)$. Since a and A_a are Turing equivalent, we have $A_a \notin M$, which allows us to apply Lemma 5.5, the Main Lemma. By hypothesis, $\mathcal{P}^M(\mathbb{H}^M)$ is countable, so fix an enumeration $\langle D_n \in \mathcal{P}^M(\mathbb{H}^M) : n \in \omega\rangle$ of the dense subsets of $\mathbb{H}^M$ in M.

We will construct a generic G for $\mathbb{H}^M$ over M. Let $\dot x$ be the canonical name for x. The forcing extension will be $M[G] = M[x]$.

First, apply Lemma 5.5 to get $p_0 \leq^{A_a} 1$ such that $p_0 \in D_0$. Next, apply Lemma 5.5 to get $p_0' \leq^{A_a} p_0$ and $m_0 \in \omega$ such that

$$p_0' \Vdash \dot\tau(\dot x)(0) = \check m_0.$$

Now we have that $p_0' \leq^{A_a} p_0 \leq^{A_a} 1$ and so we have that none of the numbers on the stem of p_0' are elements of A. That is, p_0' has "not hit A yet", and so our final value of $f_a(x)(0)$ can be anything. Now if we let $k \in A$, we can extend p_0' so that the new stem is $\mathrm{Stem}(p_0')^\frown k$ and this will define $f_a(x)(0)$. So, extend the stem of p_0' by one to get $p_0'' \leq p_0'$ in a way to ensure that $f_a(x)(0) = m_0$.

Next, apply Lemma 5.5 to get $p_1 \leq^{A_a} p_0''$ such that $p_1 \in D_1$. Next, apply Lemma 5.5 to get $p_1' \leq^{A_a} p_1$ and $m_1 \in \omega$ such that

$$p_1' \Vdash \dot\tau(\dot x)(1) = \check m_1.$$

Next, extend the stem of p_1' by one to get $p_1'' \leq p_1'$ in a way to ensure that $f_a(x)(1) = m_1$.

Continue like this infinitely. Since we have constructed a generic G over M, we have that for each $i < \omega$,

$$M[x] \models \dot\tau_G(x)(i) = m_i.$$

So by the hypothesis on $\dot\tau$, we have

$$g(x)(i) = m_i$$

for all i. On the other hand, we have ensured that for each $i < \omega$,

$$f_a(x)(i) = m_i.$$

Thus, $f_a(x) = g(x)$. Hence, f_a and g do not have disjoint graphs, which is what we wanted to show. $\square$

7. $\{f_a^{GC} : a \in {}^\omega\omega\}$ Cannot Be Avoided In The Solovay Model

In this section we will show that $\{f_a^{GC} : a \in {}^\omega\omega\}$ cannot be avoided in the Solovay model.

Theorem 7.1. *Let M be an inner model of ZFC and let κ be a strongly inaccessible cardinal in M. Assume $V = M[G]$ where G is generic for the Levy collapse of κ over M. Fix $C \in {}^\omega Ord$ and let $g : {}^\omega\omega \to {}^\omega\omega$ be such that there is a formula φ such that for each $x \in {}^\omega\omega$ and $n, m \in \omega$,*

$$g(x)(n) = m \Leftrightarrow \varphi(C, x, n, m).$$

Then for all $a \in {}^\omega\omega$,

$$[f_a^{GC} \cap g = \emptyset] \Rightarrow a \in M[C].$$

Proof. Given any $x \in {}^\omega\omega$, by the factoring of the Levy collapse for countable sets of ordinals (see Corollary 26.11 in [9]), V is a generic extension of $M[C, x]$ by the Levy collapse of κ, and $\omega_1 = \kappa$ is inaccessible in $M[C, x]$. Since the Levy collapse is homogeneous, for any x, n, m we have

$$\varphi(C, x, n, m) \Leftrightarrow M[C, x] \models 1 \Vdash \varphi(\check{C}, \check{x}, \check{n}, \check{m}).$$

Letting $\tilde{\varphi}(C, x, n, m)$ be the formula $1 \Vdash \varphi(\check{C}, \check{x}, \check{n}, \check{m})$, we have

$$g(x)(n) = m \Leftrightarrow M[C, x] \models \tilde{\varphi}(C, x, n, m).$$

This shows that $M[C]$ can understand g on its forcing extensions by the $\mathbb{H}^{M[C]}$ forcing. Note also that $\mathcal{P}^{M[C]}(\mathbb{H}^{M[C]})$ is countable, because $\omega_1 = \kappa$ is inaccessible in $M[C]$. We can now quote Theorem 6.1 using the model $M[C]$ and we are done. $\square$

Note that in the theorem above, $M[C] \cap {}^\omega\omega$ is countable, so g can be disjoint from only countably many of the f_a^{GC} functions.

Corollary 7.2. *Let κ be an inaccessible cardinal. Let G be generic for the Levy collapse of κ over V. Then*

$$\mathrm{HOD}({}^\omega \mathit{Ord})^{V[G]} \models (\forall g : {}^\omega\omega \to {}^\omega\omega)\, g \textit{ cannot avoid } \{f_a^{GC} : a \in {}^\omega\omega\}.$$

8. AD^+ Implies $\{f_a^{GC} : a \in {}^\omega\omega\}$ Cannot Be Avoided

We showed that ZF+PSP implies Ψ in Section 3. Thus a consistency strength upper bound for ZF + Ψ is one inaccessible cardinal, because the PSP holds in the Solovay model. Also, AD implies the PSP, so Ψ holds in all models of AD (and hence in all models of AD^+). The way we showed that the PSP implies Ψ is by showing that the PSP implies that $\{f_a^{PSP} : a \in {}^\omega\omega\}$ cannot be avoided.

The point of this section is to show that AD^+ implies that $\{f_a^{GC} : a \in {}^\omega\omega\}$ cannot be avoided. Previously in Section 7 we showed that $\{f_a^{GC} : a \in {}^\omega\omega\}$ cannot be avoided in the Solovay model.

We will use the following well known fact.

Fact 8.1. *Assume there is no injection from ω_1 into ${}^\omega\omega$. Let M be an inner model of ZFC. Then ω_1^V is a strong limit cardinal in M. Since $\beth_\omega$ is the first strong limit cardinal and $|\mathbb{H}| < \beth_\omega$, it follows that $\mathcal{P}^M(\mathbb{H}^M)$ is countable (in V).*

Theorem 8.2. *Assume there is no injection from ω_1 into ${}^\omega\omega$. Let $g : {}^\omega\omega \to {}^\omega\omega$ be ∞-Borel with code $C \subseteq Ord$. Then for all $a \in {}^\omega\omega$,*

$$[f_a^{GC} \cap g = \emptyset] \Rightarrow a \in L[C].$$

Hence, g cannot avoid $\{f_a^{GC} : a \in {}^\omega\omega\}$.

Proof. Use Theorem 6.1 with $M = L[C]$. To see that the hypotheses are satisfied, note that by the nature of ∞-Borel codes, M can understand g on all of its forcing extensions. □

Corollary 8.3. *Assume AD^+. No function $g : {}^\omega\omega \to {}^\omega\omega$ can avoid $\{f_a^{GC} : a \in {}^\omega\omega\}$.*

Proof. AD^+ implies that every set of reals is ∞-Borel, and hence that every $g : {}^\omega\omega \to {}^\omega\omega$ is ∞-Borel. Also AD^+ implies AD, which in turn implies there is no injection of ω_1 into ${}^\omega\omega$. □

9. Comparing f_a^{PSP} and f_a^{GC} for Projective g

In [7] we showed the following, by considering ω-models of ZF and by showing the contrapositive:

Theorem 9.1. *Fix $c \in {}^\omega\omega$. Let $g : {}^\omega\omega \to {}^\omega\omega$ be $\Delta^1_1(c)$. Then for any $a \in {}^\omega\omega$,*

$$[f_a^{GC} \cap g = \emptyset] \Rightarrow a \in \Delta^1_1(c).$$

Moving up the definability hierarchy to $\Delta^1_2(c)$ functions g, we showed the following. We will sketch the proof for reference.

Theorem 9.2. *Fix $c \in {}^\omega\omega$. Assume ω_1 is inaccessible in $L[c]$. Let $g : {}^\omega\omega \to {}^\omega\omega$ be $\Delta^1_2(c)$. Then for any $a \in {}^\omega\omega$,*

$$[f^{GC}_a \cap g = \emptyset] \Rightarrow a \in L[c].$$

Proof. Assume that $a \notin L[c]$. We will show that $g \cap f^{GC}_a \neq \emptyset$. By the Shoenfield Absoluteness theorem, $L[c]$ can understand g on all of its forcing extensions. Thus by Theorem 6.1 we have that for any $a \in {}^\omega\omega$,

$$[f^{GC}_a \cap g = \emptyset] \Rightarrow a \in L[c],$$

which is what we want. □

We believe that the inaccessible cardinal from Theorem 9.2 can be removed (and $L[c]$ need not be countable). The assumption that ω_1 is inaccessible in $L[c]$ is only needed to get $\mathcal{P}^{L[c]}(\mathbb{H}^{L[c]})$ to be countable. However, we can always force it to be countable and then we can attempt to use the Shoenfield absoluteness theorem to get what we want.

Moving up the projective hierarchy, in [7] we showed the following:

Theorem 9.3. *Fix $c \in {}^\omega\omega$. Assume PD (Projective Determinacy). Let $g : {}^\omega\omega \to {}^\omega\omega$ be $\Delta^1_n(c)$ for some $n \geq 3$. Then*

$$[f^{GC}_a \cap g = \emptyset] \Rightarrow a \in \mathcal{M}_{n-2}(c).$$

The proof of Theorem 9.3 uses that $\mathcal{M}_{n-2}(c)$ exists, that ω_1 is inaccessible in this model, and that its forcing extensions by Tree-Hechler Forcing $\mathbb{H}$ can *compute* $\Sigma^1_n(c)$ truth. Here, $\mathcal{M}_n(c)$ is a canonical inner model with n Woodin cardinals and containing c. The requirement that ω_1 be inaccessible is only needed to get the collection of dense subsets of $\mathbb{H}$ in the inner model to be countable in V.

Note that assuming PD, we have that a is Δ^1_2 in c and a countable ordinal iff $a \in L[c]$. For $n \geq 3$, a is Δ^1_n in c and a countable ordinal iff $a \in \mathcal{M}_{n-2}(c)$ [19]. Thus, we may succinctly write the following:

Fact 9.4. *Assume PD. Let $1 \leq n < \omega$. Let $g : {}^\omega\omega \to {}^\omega\omega$ be a $\Delta^1_n(c)$ function for some fixed $c \in {}^\omega\omega$. Then $f^{GC}_a \cap g = \emptyset$ implies a is Δ^1_n in c and a countable ordinal.*

Recall the definitions of D^{PSP}_g and D^{GC}_g from Definition 3.4. For $g : {}^\omega\omega \to {}^\omega\omega$ in a (lightface) projective pointclass Γ, the situation is recorded by the following table (assuming PD). The takeaway is that depending on the complexity of g, sometimes we have a better bound on D^{GC}_g, and other times we have a better bound on D^{PSP}_g.

For n odd, C_n is the largest countable Π^1_n set. For n even, C_n is the largest countable Σ^1_n set, which is also that set of all reals that are Δ^1_n

TABLE 1.

Γ	D_g^{GC} bound	D_g^{PSP} bound
Δ^1_1	$D_g^{GC} \subseteq Q_1$	$D_g^{PSP} \subseteq C_1$
Δ^1_2	$D_g^{GC} \subseteq C_2$	$D_g^{PSP} \subseteq C_3$
Δ^1_3	$D_g^{GC} \subseteq Q_3$	$D_g^{PSP} \subseteq C_3$
Δ^1_4	$D_g^{GC} \subseteq C_4$	$D_g^{PSP} \subseteq C_5$
Δ^1_5	$D_g^{GC} \subseteq Q_5$	$D_g^{PSP} \subseteq C_5$
...	...	...

in a countable ordinal. For n odd, $Q_n \supseteq C_n$ is the set of all reals that are Δ^1_n in a countable ordinal. The middle column of the table is by Theorems 9.1, 9.2, 9.3.

The rightmost column comes from the following argument: Suppose n is odd for simplicity. Suppose g is in the pointclass Δ^1_n. So g is Σ^1_n. The set D_g^{PSP} is $\forall^{\mathbb{R}}\neg\Sigma^1_n = \Pi^1_n$ by Lemma 3.5. Thus if the pointclass Π^1_n has the PSP and D_g^{PSP} is not contained in the largest countable Π^1_n set, then D_g^{PSP} must have a perfect subset. By PD we have that Π^1_n has the PSP and since D_g^{PSP} does not contain a perfect subset, D_g^{PSP} must be contained in the largest countable Π^1_n set (which is C_n).

Note that we can combine the f_a^{GC} and f_a^{PSP} families together into one to get the best of both worlds:

Definition 9.5. Let $\{f_a^{BOTH} : a \in {}^\omega\omega\}$ be the family of functions from ${}^\omega\omega$ to ${}^\omega\omega$ such that for any reals $a, x \in {}^\omega\omega$,

$$f_a^{BOTH}(0^\frown x) = f_a^{GC}(x)$$

$$f_a^{BOTH}(1^\frown x) = f_a^{PSP}(x)$$

and for $i \geq 2$, define $f_a^{BOTH}(i^\frown x)$ to be the zero sequence.

Let $D_g^{BOTH} := \{a \in {}^\omega\omega : f_a^{BOTH} \cap g = \emptyset\}$. See Table 2 for the bounds for the D_g^{BOTH} sets for projective g:

10. FUNCTIONS IN $L(\mathbb{R})[\mathcal{U}]$

The point of this section is to show, assuming large cardinals, that functions $g : {}^\omega\omega \to {}^\omega\omega$ in models of the form $L(\mathbb{R})[\mathcal{U}]$, where $\mathcal{U}$ is a selective ultrafilter on ω, cannot avoid $\{f_a^{GC} : a \in {}^\omega\omega\}$. Additionally, we will recall that such models satisfy the PSP, and so functions in these models also cannot avoid $\{f_a^{PSP} : a \in {}^\omega\omega\}$.

The significance of Lemma 10.2 that is soon to come is that if the PSP holds in an appropriate forcing extension on $L(\mathbb{R})$, then no g in

Table 2.

Γ	D_g^{BOTH} bound	
Δ^1_1	$D_g^{BOTH} \subseteq C_1$	
Δ^1_2	$D_g^{BOTH} \subseteq C_2$	
Δ^1_3	$D_g^{BOTH} \subseteq C_3$	
Δ^1_4	$D_g^{BOTH} \subseteq C_4$	
Δ^1_5	$D_g^{BOTH} \subseteq C_5$	
...	...	...

that forcing extension can avoid $\{f_a^{GC} : a \in {}^\omega\omega\}$. Note however that Lemma 10.2 applies to the forcing over $L(\mathbb{R})$ to add a Cohen subset of ω_1. However, in that forcing extension, there is a well-ordering of $\mathbb{R}$, so Ψ fails there.

The hypothesis of the next lemma follows from a proper class of Woodin cardinals. This is because of the following fact:

Fact 10.1. *Assume* AC. *Assume there is a proper class of Woodin cardinals. Then the following hold:*

1) *Every set of reals in $L(\mathbb{R})$ is universally Baire. Moreover, for every universally Baire set $A \subseteq \mathbb{R}$, the model $L(A, \mathbb{R})$ satisfies AD^+ (Theorem 7.5 of* [21]*) and every set of reals in $L(A, \mathbb{R})$ is universally Baire (Theorem 7.4 of* [21]*).*
2) *Every universally Baire binary relation on ${}^\omega\omega$ can be uniformized by a universally Baire function* [17].

Let us briefly discuss part of 1) for the interested reader. Assume that we already have the result that a proper class of Woodin cardinals implies $\mathrm{AD}^{L(\mathbb{R})}$. Now assume that there is a proper class of Woodin cardinals. Then in every forcing extension there is a proper class of Woodin cardinals. Hence $\mathrm{AD}^{L(\mathbb{R})}$ holds in every forcing extension. This by [5] implies that every set of reals in $L(\mathbb{R})$ is universally Baire.

Lemma 10.2. *Assume that for each binary relation E on ${}^\omega\omega$ in $L(\mathbb{R})$, E has a uniformization u such that $L(u, \mathbb{R}) \models AD^+$ (this holds if there is a proper class of Woodin cardinals). Let $\mathbb{Q} \in L(\mathbb{R})$ be a forcing that does not add reals (when forcing over $L(\mathbb{R})$) and whose underlying set is ${}^\omega\omega$. Let $\dot{g} \in L(\mathbb{R})$ be such that $(1 \Vdash_{\mathbb{Q}} \dot{g} : {}^\omega\omega \to {}^\omega\omega)^{L(\mathbb{R})}$. Then there exists a set of ordinals $C \subseteq Ord$ in an inner model of AD^+ containing all the reals such that $(\forall q \in \mathbb{Q})(\forall a \in {}^\omega\omega)$*

$$(q \Vdash_{\mathbb{Q}} f_a^{GC} \cap \dot{g} = \emptyset)^{L(\mathbb{R})} \Rightarrow a \in L[C, q].$$

Proof. Since we can uniformize every binary relation on ${}^{\omega}\omega$ that is in $L(\mathbb{R})$, let $u : \mathbb{Q} \times {}^{\omega}\omega \to \mathbb{Q} \times {}^{\omega}\omega$ be such that $L(u, \mathbb{R}) \models \mathrm{AD}^+$ and $(\forall q \in \mathbb{Q})(\forall x \in {}^{\omega}\omega)$, if $u(q, x) = (q', y)$, then $q' \leq q$ and

$$(q' \Vdash_{\mathbb{Q}} \dot{g}(\check{x}) = \check{y})^{L(\mathbb{R})}.$$

Since $L(u, \mathbb{R}) \models \mathrm{AD}^+$, let (C, φ) be an ∞-Borel code for u in $L(u, \mathbb{R})$. That is, $(\forall q, q' \in \mathbb{Q})(\forall x, y \in {}^{\omega}\omega)$

$$u(q, x) = (q', y) \Leftrightarrow L[C, q, x, q', y] \models \varphi(C, q, x, q', y).$$

Note that by our convention for ∞-Borel codes for functions to ${}^{\omega}\omega$ or similar ranges, if $u(q, x) = (q', y)$, then $q', y \in L[C, q, x]$.

Now fix $q \in \mathbb{Q}$. Assume that $a \notin L[C, q]$. We will show that $\neg(q \Vdash_{\mathbb{Q}} \dot{g} \cap \check{f}_a = \emptyset)^{L(\mathbb{R})}$. We will do this by constructing a $q' \leq q$ and an $x \in {}^{\omega}\omega$ such that $(q' \Vdash_{\mathbb{Q}} \dot{g}(\check{x}) = f^{GC}_a(\check{x}))^{L(\mathbb{R})}$. Consider $L[C, q]$. The x will be generic over this model by the forcing $\mathbb{H}^{L[C,q]}$. Then, setting $(q', y) = u(q, x)$, we will have $(q' \Vdash_{\mathbb{Q}} \dot{g}(\check{x}) = \check{y})^{L(\mathbb{R})}$. At the same time, we will construct x so that $f^{GC}_a(x) = y$.

Let $\dot{x}$ be $\mathbb{H}^{L[C,q]}$-name such that $1 \Vdash \dot{x} = \bigcup\{t : (\exists h)\,(t, h) \in \dot{G}\}$, where $\dot{G}$ is the canonical name for the generic filter. That is, $\dot{x}$ is a name for the real x we will construct, where $x = \{t : (\exists h)\,(t, h) \in \dot{G}\}$ where G is the generic filter we construct. We will now construct x by building a generic filter for $\mathbb{H}^{L[C,q]}$ over $L[C, q]$. Let $\dot{q}', \dot{y} \in L[C, q]$ be such that $(1 \Vdash_{\mathbb{H}} \varphi(\check{C}, \check{q}, \dot{x}, \dot{q}', \dot{y}))^{L[C,q]}$. Then, letting $q' = (\dot{q}')_x$ and $y = (\dot{y})_x$ be the valuations of these names with respect to the generic x, we will have $L[C, q, x] \models \varphi(C, q, x, q', y)$, so $u(q, x) = (q', y)$, which implies $q' \leq q$ and $q' \Vdash_{\mathbb{Q}} (\dot{g}(\check{x}) = \check{y})^{L(\mathbb{R})}$.

Let $\langle D_i : i < \omega\rangle$ be enumeration of the dense subsets of $\mathbb{H}^{L[C,q]}$ in $L[C, q]$. Let $p_0 \leq^A 1$ be in D_0. Let $p'_0 \leq^A p_0$ and $m_0 \in \omega$ be such that p'_0 decides $\dot{y}(0)$ to be m_0. That is, $(p'_0 \Vdash_{\mathbb{H}} \check{y}(0) = \check{m}_0)^{L[C,q]}$. Let $p''_0 \leq p'_0$ extend the stem of p'_0 by one to ensure that $f^{GC}_a(x)(0) = m_0$.

Now let $p_1 \leq^A p''_0$ be in D_1. Let $p'_1 \leq^A p_1$ and $m_1 \in \omega$ be such that $(p'_1 \Vdash_{\mathbb{H}} \check{y}(1) = \check{m}_1)^{L[C,q]}$. Let $p''_1 \leq p'_1$ extend the stem of p'_1 by one to ensure that $f^{GC}_a(x)(1) = m_1$.

Continue this procedure infinitely. The descending sequence of conditions constructed yields a generic ultrafilter G for $\mathbb{H}^{L[C,q]}$. By the way $x = (\dot{x})_G$ was constructed, we have $f^{GC}_a(x) = m_i$ for all $i < \omega$. We also have $y(i) = m_i$ for all $i < \omega$. Finally, we have that $(q' \Vdash_{\mathbb{Q}} \dot{g}(\check{x}) = \check{y})^{L(\mathbb{R})}$. This completes the proof. □

Observation 10.3. *Assume that* PSP *holds. Then a forcing extension that does not add reals satisfies* PSP *iff every uncountable set of reals in the extension has an uncountable subset in the ground model. This* *is because every perfect set of reals in the extension is already in the ground model.*

We will use the *tower number* for the next lemma.

Definition 10.4. Given $A, B \subseteq \omega$, we write $A \supseteq^* B$ (and say A is a *superset mod* finite of B) iff $B - A$ is finite. A *tower* is a sequence $\langle A_\alpha : \alpha < \lambda \rangle$ of infinite subsets of ω (where λ is an ordinal) such that $(\forall \alpha < \beta < \lambda)\, A_\alpha \supseteq^* A_\beta$. The tower number $\mathfrak{t}$ is the length λ of the shortest tower that cannot be end extended to a strictly longer tower.

In other words, the tower number $\mathfrak{t}$ is the smallest length λ of a tower $\langle A_\alpha : \alpha < \lambda \rangle$ such that there is no infinite $B \subseteq \omega$ such that $(\forall \alpha < \lambda)\, A_\alpha \supseteq^* B$. See [1] for a discussion of the tower number.

Paul Larson pointed out this next argument, along with using the generic absoluteness of the theory of $L(\mathbb{R})$.

Lemma 10.5. *Assume* AC. *Assume* $\omega_1 < \mathfrak{t}$. *Let* $\mathbb{Q}$ *be the* $P(\omega)/Fin$ *forcing. Then* $(1 \Vdash_{\mathbb{Q}} \text{PSP})^{L(\mathbb{R})}$.

Proof. Fix $\dot{S} \in L(\mathbb{R})$ and q such that $(q \Vdash \dot{S} \subseteq {}^\omega\omega \text{ is uncountable})^{L(\mathbb{R})}$. We will construct a $q' \leq q$ that forces (over $L(\mathbb{R})$) that $\dot{S}$ has an uncountable subset in $L(\mathbb{R})$. By induction, construct (in V) a sequence $\langle (q_\alpha, b_\alpha) : \alpha < \omega_1 \rangle$ such that 1) the b_α's are distinct reals, 2) the q_α's are decreasing with $q \geq q_0$, and 3) $(q_\alpha \Vdash \check{b}_\alpha \in \dot{S})^{L(\mathbb{R})}$ for each $\alpha < \omega_1$. Every countable initial segment of the sequence we are constructing will be in $L(\mathbb{R})$ (because $L(\mathbb{R})$ contains every countable sequence of reals). Note that we do not get stuck at any stage, and so can construct the entire sequence. However note that the entire (length ω_1) sequence may not be in $L(\mathbb{R})$ because $L(\mathbb{R})$ may satisfy AD and hence have no injection of ω_1 into $\mathbb{R}$.

Let q' be a lower bound of the q_α's, which exists because they form a decreasing, with respect to almost inclusion, sequence of infinite subsets of ω, and this sequence cannot be maximal because $\omega_1 < \mathfrak{t}$. That is, we construct q' in V, however it must be in $L(\mathbb{R})$ because $L(\mathbb{R})$ contains all the reals. Now let $K = \{b \in {}^\omega\omega : (q' \Vdash \check{b} \in \dot{S})^{L(\mathbb{R})}\}$. Note that $K \in L(\mathbb{R})$. In V we can see that $\{b_\alpha : \alpha < \omega_1\} \subseteq K$, so K is uncountable (in V). But then also K is uncountable in $L(\mathbb{R})$. Now note that $(q' \Vdash \check{K} \subseteq \dot{S})^{L(\mathbb{R})}$. Also note that $(q' \Vdash \check{K} \text{ is uncountable})^{L(\mathbb{R})}$ because the forcing does not add any new countable sequences of reals. Hence q' forces that $\dot{S}$ has an uncountable subset that is in the ground model $L(\mathbb{R})$, which by the observation above finishes the proof. □

Theorem 10.6. *Assume* AC. *Assume there is a proper class of Woodin cardinals. Let* $\mathcal{U}$ *be a selective ultrafilter on* ω. *Let* $g : {}^\omega\omega \to {}^\omega\omega$ *be in* $L(\mathbb{R})[\mathcal{U}]$. *Then* g *cannot avoid* $\{f_a^{GC} : a \in {}^\omega\omega\}$.

Proof. Let $\mathbb{Q}$ be the $P(\omega)/\mathrm{Fin}$ forcing. Since there is a proper class of Woodin cardinals, the first order theory of $L(\mathbb{R})$ cannot be changed by any set sized forcing (see Theorem 7.22 of [20], also Theorem 3.1.12 in [12]). There is a forcing extension of V in which $\omega_1 < \mathfrak{t}$. By Lemma 10.5, in that forcing extension we have $(1 \Vdash_{\mathbb{Q}} \mathrm{PSP})^{L(\mathbb{R})}$. Thus, in V we have $(1 \Vdash_{\mathbb{Q}} \mathrm{PSP})^{L(\mathbb{R})}$.

Another consequence of a proper class of Woodin cardinals is that an ultrafilter on ω is selective iff it is $\mathbb{Q}$-generic over $L(\mathbb{R})$ (see [4] and [10]). Thus, we will show that every name $\dot{g} \in L(\mathbb{R})$ for a function from ${}^\omega\omega$ to ${}^\omega\omega$ satisfies

$$L(\mathbb{R}) \models 1 \Vdash_{\mathbb{Q}} \dot{g} \text{ cannot avoid } \{f_a^{GC} : a \in {}^\omega\omega\}.$$

Towards a contradiction, fix $\dot{g} \in L(\mathbb{R})$ and $q \in \mathbb{Q}$ such that

$$L(\mathbb{R}) \models q \Vdash_{\mathbb{Q}} \{a : \dot{g} \cap \check{f}_a = \emptyset\} \text{ is uncountable.}$$

Since $L(\mathbb{R}) \models q \Vdash_{\mathbb{Q}} \mathrm{PSP}$, by the observation above fix a condition $q' \leq q$ and an uncountable set $S \subseteq {}^\omega\omega$ in $L(\mathbb{R})$ such that for all $a \in S$,

$$L(\mathbb{R}) \models q' \Vdash_{\mathbb{Q}} [\dot{g} \cap \check{f}_a = \emptyset].$$

Since there is a proper class of Woodin cardinals, Apply Lemma 10.2 to get the $C \subseteq \mathrm{Ord}$ described there. We have

$$(\forall a \in S)\, a \in L[C, q'],$$

which is a contradiction because since $L[C, q']$ is an inner model of ZFC inside a model of AD, ${}^\omega\omega \cap L[C, q']$ is countable. □

11. Final Questions

We close with a few questions.

Question 11.1. Does AD imply that no $g : {}^\omega\omega \to {}^\omega\omega$ can avoid $\{f_a^{GC} : a \in {}^\omega\omega\}$?

More generally, we can ask the following:

Question 11.2. Does PSP imply that no $g : {}^\omega\omega \to {}^\omega\omega$ can avoid $\{f_a^{GC} : a \in {}^\omega\omega\}$?

12. Acknowledgements

I would like the thank Andreas Blass, Paul Larson, Grigor Sargsyan, and Trevor Wilson for discussions on this project. Larson pointed out the arguments for Proposition 2.2 and Lemma 10.5. He also verified that a proper class of Woodin cardinals implies every relation on ${}^{\omega}\omega$ in $L(\mathbb{R})$ can be uniformized in some model of AD^+. Wilson explained how much truth small forcing extensions of $\mathcal{M}_n(c)$ can compute. He also explained how PD suffices for Fact 9.4, instead of ω Woodin cardinals. Sargsyan explained how to show that countable sets or reals are well-behaved using sufficient determinacy axioms. We also thank the referee for many useful suggestions.

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