

THE HALPERN-LÄUCHLI THEOREM AT A MEASURABLE CARDINAL

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ABSTRACT. Several variants of the Halpern-Läuchli Theorem for trees of uncountable height are investigated. For κ weakly compact, we prove that the various statements are all equivalent. We show that the strong tree version holds for one tree on any infinite cardinal. For any finite $d \geq 2$, we prove the consistency of the Halpern-Läuchli Theorem on d many normal κ -trees at a measurable cardinal κ , given the consistency of a $\kappa + d$ -strong cardinal. This follows from a more general consistency result at measurable κ , which includes the possibility of infinitely many trees, assuming partition relations which hold in models of AD.

1. INTRODUCTION

Halpern and Läuchli proved their celebrated theorem regarding Ramsey theory on products of trees in [12] as a necessary step for their construction in [13] of a model of ZF in which the Boolean Prime Ideal Theorem holds but the Axiom of Choice fails. Since then, many variations have been established and applied. One of the earliest of these is due to Milliken, who extended the Halpern-Läuchli Theorem to colorings of finite products of strong trees in [16]. As the two versions are equivalent, this version is often used synonymously with Halpern and Läuchli's original version. Milliken further proved in [17] that the collection of strong trees forms, in modern terminology, a topological Ramsey space. Further variations and applications include ω many perfect trees in [15]; partitions of products in [2]; the density version in [5]; the dual version in [21]; canonical equivalence relations on finite strong trees in [22]; and applications to colorings of subsets of the rationals in [1] and [23], and to finite Ramsey degrees of the Rado graph in [18] which in turn was applied to show the Rado graph has the rainbow Ramsey property in [4], to name just a few.

Shelah produced the first result generalizing the Halpern-Läuchli Theorem to an uncountable height tree in [19]. There, in Lemma 4.1, Shelah proved that for certain models of ZFC with a measurable cardinal κ , given any finite m and any coloring of the m -sized subsets of the levels of the tree ${}^{<\kappa}2$, (that is, a coloring of $\bigcup_{\zeta < \kappa} [{}^{\zeta}2]^m$), into less than κ many colors, there is a strong subtree (see Definition 2.1) $T \subseteq {}^{<\kappa}2$ on which the coloring takes only finitely many colors. Shelah's proof builds on and extends Harrington's forcing argument for the Halpern-Läuchli Theorem on ${}^{<\omega}2$, and assumes that κ is a cardinal which is measurable after adding λ many Cohen

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subsets of κ , where λ is large enough that the partition relation $\lambda \rightarrow (\kappa^+)_2^{2^m}$ holds. Thus, his result also holds for any κ which is supercompact after a Laver treatment.

In [6], Dzamonja, Larson and Mitchell extended Shelah's proof to include colorings of all antichains in the tree ${}^{<\kappa}2$ of a fixed finite size m , rather than just colorings of subsets of the same levels. They proved that given a coloring of m -sized antichains of the tree ${}^{<\kappa}2$ with less than κ many colors, there is a strong subtree T isomorphic to ${}^{<\kappa}2$ on which the set of m -sized antichains takes on only finitely many colors. These finitely many colors are classified by the embedding types of the trees induced by the antichains. They then applied this result to find the Ramsey degrees of $< \kappa$ colorings of $[\mathbb{Q}_\kappa]^m$, m any finite integer, and of $< \kappa$ colorings of the copies of a fixed finite graph inside the Rado graph on κ many vertices in [6] and [7], respectively. Here \mathbb{Q}_κ is the unique, up to isomorphism, κ -dense linear order of size κ . A poset $\langle L, <_L \rangle$ is κ -dense iff whenever $A, B \subseteq L$ are such that $a < b$ for all $a \in A$ and $b \in B$, then there is some $c \in L$ such that $a < c < b$ for all $a \in A$ and $b \in B$.

This work has left open the following questions.

Question 1.1. For which uncountable cardinals κ can the Halpern-Läuchli Theorem hold for trees of height κ , either in ZFC or consistently?

Question 1.2. What is the consistency strength of the Halpern-Läuchli Theorem for κ uncountable; in particular for κ a measurable cardinal?

Question 1.3. Given a fixed number of trees, are the weaker (somewhere dense) and stronger (strong tree) forms of the Halpern-Läuchli Theorem equivalent, for κ uncountable?

In Section 3, we answer Question 1.3 for the case of weakly compact cardinals: Theorem 3.7 shows that when κ is weakly compact, all the various weaker and stronger forms of the Halpern-Läuchli Theorem on δ many regular κ trees (see Section 2) are equivalent, where δ can be any cardinal less than κ . In Fact 2.4, we show that the somewhere dense version for finitary colorings on one tree on any cardinal is a ZFC result. In fact, if κ is an infinite cardinal, then the strong tree version of the Halpern-Läuchli Theorem holds for finitary colorings of one regular tree on κ , thus answering Question 1.1 for the case of one tree and finitely many colors.

In Section 4, for any fixed positive integer d , Theorem 4.7 shows that it is consistent for the Halpern-Läuchli Theorem to hold on d many regular trees on a measurable cardinal κ . Further, this theorem yields that a $\kappa + d$ -strong cardinal is an upper bound for the consistency strength of the Halpern-Läuchli Theorem for d many regular trees on a measurable cardinal κ , thus partially answering Questions 1.1 and 1.2. This is weaker than the upper bound of a cardinal κ which is $(\kappa + 2d)$ -strong, that would be needed if one simply lifted the standard version of Harrington's forcing proof to a measurable cardinal. This result also presents an interesting contrast to the $(\kappa + 2d + 2)$ -strong cardinal mentioned as an upper bound for the consistency strength of Lemma 4.1 in [19] and its strengthening Theorem 2.5 in [6], which colors antichains of size d in one tree on κ . We conjecture in Section 5 that the consistency strength of the Halpern-Lauchli Theorem on d many regular trees for κ measurable is in fact a $\kappa + d$ -strong cardinal. Theorem 4.7 follows from a more general result, Theorem 4.4, which includes the possibility of infinitely many

trees assuming certain partition relations which hold assuming AD. Section 5 concludes with open problems and conjectures and their relationships with previous work.

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2. VARIANTS OF THE HALPERN-LÄUCHLI THEOREM AND SIMPLE IMPLICATIONS

This section contains the relevant definitions, various versions of the Halpern-Läuchli Theorem, and the immediate implications between them. A tree $T \subseteq {}^{<\kappa}\kappa$ is a κ -tree if T has cardinality κ and every level of T has cardinality less than κ . T is *perfect* if for each node t in T there is an extension of t which splits in T . We shall call a tree $T \subseteq {}^{<\kappa}\kappa$ *regular* if it is a perfect κ -tree in which every maximal branch has cofinality κ . Given any subset $T \subseteq {}^{<\kappa}\kappa$ and $\zeta < \kappa$, we write

$$T(\zeta) := T \cap {}^\zeta\kappa$$

for the set of nodes on the ζ -th level of T . Given $t \in T$,

$$T[t] := \{s \in T : s \sqsubseteq t \text{ or } t \sqsubseteq s\}.$$

In this paper, the successor of a node always means immediate successor.

Next, we define the notion of strong subtree, originally defined by Milliken in [16].

Definition 2.1. Let $T \subseteq {}^{<\kappa}\kappa$ be regular. A tree $T' \subseteq T$ is a *strong subtree* of T as witnessed by some set $A \subseteq \kappa$ cofinal in κ if and only if T' is regular and for each $t \in T'(\zeta)$ for $\zeta < \kappa$, $\zeta \notin A$ implies that t has a unique successor in T' on level $\zeta + 1$, and $\zeta \in A$ implies every successor of t in T is also in T' .

Given an ordinal $\delta > 0$ and a sequence $\langle X_i \subseteq {}^{<\kappa}\kappa : i < \delta \rangle$, define

$$\bigotimes_{i < \delta} X_i := \{ \langle x_i : i < \delta \rangle : (\exists \zeta < \kappa) (\forall i < \delta) x_i \in X_i(\zeta) \},$$

the *level-product* of the X_i 's. We will call a set of nodes all on the same level a *level set*. Similarly, we will call a sequence of nodes or a sequence of sets of nodes a *level sequence* if all nodes are on the same level.

The following is the strong tree version of the Halpern-Läuchli Theorem, which we shall denote by $\text{HL}(\delta, \sigma, \kappa)$.

Definition 2.2. For $\delta, \sigma > 0$ ordinals and κ an infinite cardinal, $\text{HL}(\delta, \sigma, \kappa)$ is the following statement: Given any sequence $\langle T_i \subseteq {}^{<\kappa}\kappa : i < \delta \rangle$ of regular trees and a coloring $c : \bigotimes_{i < \delta} T_i \rightarrow \sigma$, there exists a sequence of trees $\langle T'_i : i < \delta \rangle$ such that

- (1) each T'_i is a strong subtree of T_i as witnessed by the same set $A \subseteq \kappa$ independent of i , and
- (2) there is some $\sigma' < \sigma$ such that for each $\zeta \in A$, $c \upharpoonright \bigotimes_{i < \delta} T'_i(\zeta) = \{\sigma'\}$.

Given $t \in {}^{<\kappa}\kappa$, we define

$$\text{Cone}(t) := \{s \in {}^{<\kappa}\kappa : s \sqsupseteq t\}.$$

We say that $X \subseteq {}^{<\kappa}\kappa$ *dominates* $Y \subseteq {}^{<\kappa}\kappa$ if and only if

$$(\forall y \in Y)(\exists x \in X) y \sqsubseteq x.$$

We say $X \subseteq {}^{<\kappa}\kappa$ *dominates* $y \in {}^{<\kappa}\kappa$ just when X dominates $\{y\}$. We now give the definition of the Somewhere-Dense Halpern-Läuchli Theorem.

Definition 2.3. For nonzero ordinals δ and σ , and an infinite cardinal κ , $\text{SDHL}(\delta, \sigma, \kappa)$ is the statement that given any sequence $\langle T_i \subseteq {}^{<\kappa}\kappa : i < \delta \rangle$ of regular trees and any coloring

$$c : \bigotimes_{i < \delta} T_i \rightarrow \sigma,$$

there exist $\zeta < \zeta' < \kappa$, $\langle t_i \in T_i(\zeta) : i < \delta \rangle$, and $\langle X_i \subseteq T_i(\zeta') : i < \delta \rangle$ such that each X_i dominates $T_i(\zeta + 1) \cap \text{Cone}(t_i)$ and

$$|c^{\text{``}} \bigotimes_{i < \delta} X_i | = 1.$$

We point out the following simple fact.

Fact 2.4. For every infinite cardinal κ and each positive integer k , $\text{SDHL}(1, k, \kappa)$ holds.

Proof. If $k = 1$, the result is, so assume $k \geq 2$. To prove $\text{SDHL}(1, k, \kappa)$, let T be a regular subtree of ${}^{<\kappa}\kappa$ and c be a coloring from the nodes in T into k . If there exist a node $t \in T$ and a level set $X \subseteq T$ dominating the successors of t such that every node in X has color 0, then we are done. Otherwise, for each node $t \in T$ and each level set $X \subseteq T$ dominating the successors of t , $c^{\text{``}}X \cap \{1, \dots, k-1\} \neq \emptyset$. If there exist a node $t \in T$ and a level set $X \subseteq T$ dominating the successors of t such that every node in X has color 1, then we are done. Otherwise, for each node $t \in T$ and each level set $X \subseteq T$ dominating the successors of t , $c^{\text{``}}X \cap \{2, \dots, k-1\} \neq \emptyset$. Continuing in this manner, either there is a $j < k-1$ and some node t with a level set X dominating the successors of t each with c -color j , or else, for each node $t \in T$ and each level set $X \subseteq T$ dominating the successors of t , $c^{\text{``}}X \cap \{k-1\} \neq \emptyset$. In this case, choose any node $t \in T$ and list the successors of t in T as $\langle s_i : i < \eta \rangle$, where $\eta < \kappa$. For each $i < \eta$, let Y_i denote the set of all the successors of s_i . Then Y_i forms a level set dominating the successors of s_i , so $c^{\text{``}}Y_i \cap \{k-1\} \neq \emptyset$. For each $i < \eta$, take one $t_i \in Y_i$ such that $c(t_i) = k-1$. Then the set $\{t_i : i < \eta\}$ is a level set dominating every successor of t and is monochromatic in color $k-1$. \square

Definition 2.5. Given a regular tree T and a level ζ less than the height of T , a set $X \subseteq T$ is ζ -*dense* if and only if X dominates $T(\zeta)$. Given a sequence of regular trees $\langle T_i : i < \delta \rangle$ and $\vec{x} = \langle x_i : i < \delta \rangle \in \bigotimes_{i < \delta} T_i$, a sequence of sets $X_i \subseteq T_i$ for $i < \delta$ is ζ - \vec{x} -*dense* if and only if for each $i < \delta$, X_i dominates $T_i(\zeta) \cap \text{Cone}(x_i)$.

The definition of $\text{SDHL}(\delta, \sigma, \kappa)$ can be weakened by not requiring the sets X_i for $i < \delta$ to be level sets; in this case the colorings must color the full product $\prod_{i < \delta} T_i$. Call this weakening $\text{SDHL}'(\delta, \sigma, \kappa)$. A *somewhere dense matrix* for a sequence of regular trees $\langle T_i : i < \delta \rangle$ is a sequence of sets $\langle X_i \subseteq T_i : i < \delta \rangle$ such that there are nodes $t_i \in T_i$ for $i < \delta$ all of the same length such that each X_i dominates all successors of t_i in T_i . Thus, $\text{SDHL}'(\delta, \sigma, \kappa)$ is the statement that any coloring $c : \prod_{i < \delta} T_i \rightarrow \sigma$ is constant on $\prod_{i < \delta} X_i$ for some somewhere dense matrix $\langle X_i \subseteq T_i : i < \delta \rangle$. Certainly $\text{SDHL}(\delta, \sigma, \kappa)$ implies $\text{SDHL}'(\delta, \sigma, \kappa)$. Our definition of a somewhere dense matrix is not exactly the same as in [20], but both versions yield equivalent definitions of $\text{SDHL}'(\delta, \sigma, \kappa)$, which is the point of the next remark.

Remark 2.6. The $\text{SDHL}'(\delta, \sigma, \kappa)$ is in fact equivalent to the statement obtained by making the following modification. The modification is to allow the nodes t_i to come from different levels of the T_i , while requiring that the supremum η of the length of the members of $\{t_i : i < \delta\}$ to be strictly less than the length of any member of $\bigcup\{X_i : i < \delta\}$, and requiring each X_i to dominate $\text{Cone}(t_i) \cap T(\eta + 1)$ instead of the set of successors of t_i . It is routine to check that the statement under this modification is equivalent to $\text{SDHL}'(\delta, \sigma, \kappa)$.

By the definitions and previous discussions, the following implications are immediate.

Fact 2.7. *The following statements are arranged from strongest to weakest, where δ and σ are assumed to be strictly less than κ .*

- (1) $HL(\delta, \sigma, \kappa)$;
- (2) $SDHL(\delta, \sigma, \kappa)$;
- (3) $SDHL'(\delta, \sigma, \kappa)$;
- (4) $SDHL'(\delta, \sigma, \kappa)$ where the t_i need not be from the same level.

The remark above shows that in fact (4) implies (3). In the next section will show that (3) implies (1). Thus, all four statements are equivalent.

3. THE VARIOUS FORMS OF HALPERN-LÄUCHLI ARE EQUIVALENT, FOR κ WEAKLY COMPACT

This section provides proofs that all the versions of the Halpern-Läuchli Theorem stated in the previous section are equivalent, provided that κ is weakly compact.

First, we prove that if κ is weakly compact and $\delta, \sigma < \kappa$, then $\text{SDHL}'(\delta, \sigma, \kappa)$ implies $\text{SDHL}(\delta, \sigma, \kappa)$. The proof proceeds via the next three lemmas. Recall that a cardinal κ is *weakly compact* if κ is strongly inaccessible and satisfies the Tree Property at κ ; that is, every κ -tree has a cofinal branch.

Lemma 3.1. *Suppose κ is weakly compact. Let $\delta, \sigma < \kappa$ be non-zero ordinals, and let $\langle T_i \subseteq {}^{<\kappa}\kappa : i < \delta \rangle$ be a sequence of regular trees. Let U be the tree of partial colorings of $\prod_{i < \delta} T_i$ into σ many colors, where a node of U on level α corresponds to a coloring of all tuples $\langle t_i \in T_i : i < \delta \rangle$ where $\sup\{\text{lh}(t_i) : i < \delta\} < \alpha$.*

Suppose $Z \subseteq U$ has size κ and each $z \in Z$ corresponds to a coloring c_z which is not monochromatic on any somewhere dense matrix contained in its domain. Then there is a coloring c of all of $\prod_{i < \delta} T_i$ such that c is not monochromatic on any somewhere dense matrix.

Proof. Note that if $\langle u_\alpha : \alpha < \beta \rangle$ is a strictly increasing sequence of nodes of U for some limit ordinal β , then there is a unique node on level $\sup_{\alpha < \beta} \text{lh}(u_\alpha)$ that extends each u_α . Furthermore, note that U is a κ -tree. Fix a set $Z \subseteq U$ as in the hypotheses.

Let S be the set of all predecessors of elements of Z . Then S is a κ -tree. By the weak compactness of κ , fix a length κ branch through S . This branch corresponds to a coloring c of all of $\prod_{i < \delta} T_i$. Now, for each $\alpha < \kappa$ there is a node $z \in Z$ such that the domain of c_z includes $\prod_{i < \delta} \bigcup_{\zeta < \alpha} T_i(\zeta)$ and both c and c_z color those sequences the same way. For each $\alpha < \kappa$, choose one such node and label it z_α .

Now, pick any somewhere dense $\langle X_i \subseteq T_i : i < \delta \rangle$ such that there is some $\alpha < \kappa$ satisfying $(\forall i < \delta)(\forall x \in X_i) \text{lh}(x) < \alpha$. Fix such an $\alpha < \kappa$. We must show that

$|c^{\text{“}} \prod_{i < \delta} X_i | > 1$. We have that for all $\langle x_i \in X_i : i < \delta \rangle$,

$$c(\langle x_i : i < \delta \rangle) = c_{z_\alpha}(\langle x_i : i < \delta \rangle).$$

By the hypothesis on c_{z_α} , we have $|c_{z_\alpha}^{\text{“}} \prod_{i < \delta} X_i | > 1$. Thus, $|c^{\text{“}} \prod_{i < \delta} X_i | > 1$ as desired. \square

Lemma 3.2. *Suppose κ is weakly compact. Let $\sigma, \delta > 0$ be ordinals and assume $\delta, \sigma < \kappa$. Assume $SDHL'(\delta, \sigma, \kappa)$ holds. Let $\langle T_i \subseteq {}^{<\kappa}\kappa : i < \delta \rangle$ be a sequence of regular trees. Then there is a level $1 \leq \zeta < \kappa$ such that for every coloring*

$$c : \prod_{i < \delta} \bigcup_{\xi < \zeta} T_i(\xi) \rightarrow \sigma,$$

there is a somewhere dense matrix $\langle X_i \subseteq \bigcup_{\xi < \zeta} T_i(\xi) : i < \delta \rangle$ such that c is constant on $\prod_{i < \delta} X_i$.

Proof. Assume there is no such level ζ . Then for each $\zeta < \kappa$ we may pick a coloring $c_\zeta : \prod_{i < \delta} \bigcup_{\xi < \zeta} T_i(\xi) \rightarrow \sigma$ which is not constant on any somewhere dense matrix in the domain of c_ζ . Letting U be the tree of partial colorings described in Lemma 3.1, the set $Z = \{c_\zeta : \zeta < \kappa\}$ is a subset of U of size κ . Applying Lemma 3.1, we obtain a coloring $c : \prod_{i < \delta} T_i \rightarrow \sigma$ for which there is no somewhere dense matrix $\langle X_i \subseteq T_i : i < \delta \rangle$ such that $|c^{\text{“}} \prod_{i < \delta} X_i | = 1$. Hence, $SDHL'(\delta, \sigma, \kappa)$ fails, which is a contradiction. \square

The following bounded version is the analogue of the Finite Halpern-Läuchli Theorem for finitely many trees on ω (see, for instance, Theorem 3.9 in [20]).

Lemma 3.3. *Suppose κ is weakly compact. Let $\sigma, \delta < \kappa$ be nonzero ordinals and assume $SDHL'(\delta, \sigma, \kappa)$ holds. Let $\langle T_i \subseteq {}^{<\kappa}\kappa : i < \delta \rangle$ be a sequence of regular trees. Then there is a level $1 \leq \zeta < \kappa$ such that for every coloring*

$$\bar{c} : \bigotimes_{i < \delta} T_i(\zeta) \rightarrow \sigma,$$

there is a somewhere dense matrix $\langle Y_i \subseteq T_i(\zeta) : i < \delta \rangle$ such that \bar{c} is constant on $\bigotimes_{i < \delta} Y_i$.

Proof. Fix some $\zeta < \kappa$ for which the Lemma 3.2 holds. For each $i < \delta$ and each node $t \in \bigcup_{\xi < \zeta} T_i(\xi)$, associate a node $f_i(t) \in T_i(\zeta)$ such that $f_i(t) \sqsupseteq t$. Now fix a coloring $\bar{c} : \bigotimes_{i < \delta} T_i(\zeta) \rightarrow \sigma$. Let $c : \prod_{i < \delta} \bigcup_{\xi < \zeta} T_i(\xi) \rightarrow \sigma$ be defined by

$$c(\langle t_i : i < \delta \rangle) := \bar{c}(\langle f_i(t_i) : i < \delta \rangle).$$

By the property of ζ , there is a somewhere dense matrix $\langle X_i \subseteq \bigcup_{\xi < \zeta} T_i(\xi) : i < \delta \rangle$ such that $|c^{\text{“}} \prod_{i < \delta} X_i | = 1$. For each $i < \delta$, let

$$Y_i := \{f_i(t) : t \in X_i\}.$$

The sequence $\langle Y_i \subseteq T_i(\zeta) : i < \delta \rangle$ is a somewhere dense matrix, and by the definition of c we have $|\bar{c}^{\text{“}} \bigotimes_{i < \delta} Y_i | = 1$. \square

Remark 3.4. We point out that the set of $\zeta < \kappa$ satisfying Lemma 3.2 is closed upwards, and hence Lemma 3.3 also holds for ζ in a final segment of κ .

Theorem 3.5. *Suppose κ is weakly compact. Let $\sigma, \delta > 0$ be ordinals and assume $\delta, \sigma < \kappa$. Then $SDHL'(\delta, \sigma, \kappa)$ holds if and only if $SDHL(\delta, \sigma, \kappa)$ holds.*

Proof. This follows by the lemmas above. \square

Jing Zhang recognized that the proof that SDHL implies HL in [20] works for an arbitrary inaccessible κ , even one that is not weakly compact.

Lemma 3.6. *Let κ be a cardinal and let $\delta, \sigma < \kappa$ be non-zero ordinals. Let $\langle T_i : i < \delta \rangle$ be a sequence of regular trees. Fix $c : \bigotimes_{i < \delta} T_i \rightarrow \sigma$. Suppose there is a level sequence $\vec{x} = \langle x_i \in T_i : i < \delta \rangle$ such that for each $\xi < \kappa$, there is a level sequence $\langle X_i \subseteq T_i(\xi) : i < \delta \rangle$ for some ζ such that $\langle X_i : i < \delta \rangle$ is ξ - \vec{x} -dense and $|c^{\llbracket \bigotimes_{i < \delta} X_i \rrbracket}| = 1$. Then the conclusion of $HL(\delta, \sigma, \kappa)$ holds.*

Proof. We will construct a sequence of strong subtrees $\langle T'_i \subseteq T_i : i < \delta \rangle$, as witnessed by the same set $A \subseteq \kappa$ independent of i , such that for each $\zeta \in A$, there is some $\sigma_\zeta < \sigma$ such that $c^{\llbracket \bigotimes_{i < \delta} T'_i(\zeta) \rrbracket} = \{\sigma_\zeta\}$. Then, by the pigeonhole principal, there will be κ many σ_ζ that are equal to the same ordinal, call it σ' . Let $\tilde{A} \subseteq A$ be the set of levels associated to the color σ' . We may then thin each T'_i to a strong subtree T''_i as witnessed by \tilde{A} (independent of i) such that for each $\zeta \in \tilde{A}$, $c^{\llbracket \bigotimes_{i < \delta} T''_i(\zeta) \rrbracket} = \{\sigma'\}$. This is the conclusion of $HL(\delta, \sigma, \kappa)$.

For each $i < \delta$, instead of directly constructing T'_i , we will construct a subset S_i of T'_i and T'_i will be the set of all initial segments of elements of S_i . Each S_i will be the union of level sets: $S_i := \bigcup_{\zeta \in A} L_{i,\zeta}$ where each $L_{i,\zeta}$ will be a subset of $T_i(\zeta)$. At the same time, we will construct $A \subseteq \kappa$. We will have it so $(\forall \zeta \in A) |c^{\llbracket \bigotimes_{i < \delta} L_{i,\zeta} \rrbracket}| = 1$. Initially the set A is empty and no $L_{i,\zeta}$'s have been defined.

Now assume we are at some stage of the construction. There are three cases:

Case 1: No $L_{i,\zeta}$'s have been constructed so far (and A is empty). In this case, set ξ to be the level of the x_i and for each $i < \delta$, let $U_{i,\xi+1} := T_i(\xi+1) \cap \text{Cone}(x_i)$.

Case 2: Some $L_{i,\zeta}$'s have been constructed and there is a largest $\xi < \kappa$ for which some $L_{i,\xi}$ exists. Fix this ξ . For each $i < \delta$, define $U_{i,\xi+1} := T_i(\xi+1) \cap (\bigcup_{t \in L_{i,\xi}} \text{Cone}(t))$.

Case 3: The set of η 's for which the $L_{i,\eta}$'s exist is below but cofinal in some fixed $\xi < \kappa$. Let $W_i \subseteq T_i(\xi)$ be the set of $t \in T_i(\xi)$ such that the set of $\eta < \xi$ such that $t \upharpoonright \eta \in L_{i,\eta}$ is cofinal in ξ . For each $i < \delta$, define $U_{i,\xi+1} := T_i(\xi+1) \cap (\bigcup_{t \in W} \text{Cone}(t))$.

Assuming one of the three cases above holds, we have both an ordinal ξ and a set $U_{i,\xi+1} \subseteq T_i(\xi+1)$ for each $i < \delta$. Apply the hypothesis of the lemma to get a level sequence $\langle X_i : i < \delta \rangle$ that is $(\xi+1)$ - \vec{x} -dense and $|c^{\llbracket \bigotimes_{i < \delta} X_i \rrbracket}| = 1$. For each $i < \delta$, let $X'_i \subseteq X_i$ be such that each $t \in X'_i$ extends some $u \in U_{i,\xi+1}$, every $u \in U_{i,\xi+1}$ is extended by some $t \in X'_i$, and no two elements of X'_i extend the same element of $U_{i,\xi+1}$. Let ζ be the level of the X'_i . Add ζ to the set A and set $L_{i,\zeta} := X'_i$ for each $i < \delta$. This completes the construction and the proof. \square

Theorem 3.7. *Let κ be an infinite cardinal and let $\delta, \sigma < \kappa$ be non-zero ordinals. Then $SDHL(\delta, \sigma, \kappa)$ implies $HL(\delta, \sigma, \kappa)$.*

Proof. Fix $c : \bigotimes_{i < \delta} T_i \rightarrow \sigma$. Using Lemma 3.3, we claim that there is a level sequence $\vec{x} = \langle x_i \in T_i : i < \delta \rangle$ such that for each $\zeta < \kappa$, there is a level sequence $\langle X_i \subseteq T_i : i < \delta \rangle$ that is ζ - \vec{x} -dense and $|c^{\llbracket \bigotimes_{i < \delta} X_i \rrbracket}| = 1$. From this, the conclusion of $HL(\delta, \sigma, \kappa)$ follows from Lemma 3.6. To show the claim, suppose towards a contradiction that it is false. For each $\vec{x} \in \bigotimes_{i < \delta} T_i$ we can associate the first $\zeta(\vec{x}) < \kappa$ for which we cannot find a $\zeta(\vec{x})$ - \vec{x} -dense level sequence $\langle X_i : i < \delta \rangle$ such that $|c^{\llbracket \bigotimes_{i < \delta} X_i \rrbracket}| = 1$. We can build a strictly increasing sequence $\langle \eta_\alpha \in \kappa : \alpha < \kappa \rangle$

of ordinals such that for every $\alpha < \kappa$,

$$\eta_{\alpha+1} > \zeta(\vec{x}) \text{ for all } \vec{x} \in \bigcup_{\eta \leq \eta_\alpha} \bigotimes_{i < \delta} T_i(\eta).$$

For $i < \delta$, set

$$T_i^* := \bigcup_{\alpha < \kappa} T_i(\eta_\alpha).$$

Applying $\text{SDHL}(\delta, \sigma, \kappa)$ to the restriction of c to $\bigotimes_{i < \delta} T_i^*$, we find a somewhere dense matrix $\langle X_i^* : i < \delta \rangle$ of the sequence of trees $\langle T_i^* : i < \delta \rangle$ on which c is constant. This means that there exists $\alpha < \alpha' < \kappa$ and $\vec{x} = \langle x_i : i < \delta \rangle \in \bigotimes_{i < \delta} T_i(\eta_\alpha)$ such that for all $i < \delta$,

$$X_i^* \text{ dominates } \text{Cone}(x_i) \cap T_i(\eta_{\alpha'}).$$

It follows that $\langle X_i^* : i < \delta \rangle$ is an $\eta_{\alpha'}$ - \vec{x} -dense c -monochromatic level sequence, contradicting the definition of $\zeta(\vec{x})$ and the fact that

$$\eta_{\alpha'} \geq \eta_{\alpha+1} > \zeta(\vec{x}).$$

This finishes the proof. \square

Fact 2.4 and Theorem 3.7 yield the following.

Theorem 3.8. *HL(1, k, κ) holds for each positive integer k and each infinite cardinal κ .*

4. THE HALPERN-LÄUCHLI THEOREM AT A MEASURABLE CARDINAL

In Theorem 4.7, we prove that for any positive integer d , assuming the existence of a cardinal κ which is $\kappa + d$ -strong cardinal (see Definition 4.6), it is consistent that $\text{HL}(d, \sigma, \kappa)$ holds at a measurable cardinal κ , for all $\sigma < \kappa$. Actually, in Theorem 4.4 we prove an asymmetric version of the $\text{SDHL}(\delta, \sigma, \kappa)$ which holds for finitely or infinitely many trees on a measurable cardinal. The requisites for that theorem are that certain partition relations (see hypotheses 1 and 2 of Theorem 4.4) hold in a model M of ZF and that κ remains measurable after forcing over M with $\text{Add}(\kappa, \lambda)$, where $\text{Add}(\kappa, \lambda)$ denotes the forcing which adds λ many κ -Cohen subsets of κ via partial functions from $\lambda \times \kappa$ into 2 of size less than κ and λ is large enough for a certain partition relation to hold. For infinitely many trees, the relevant partition relations hold assuming AD, and hence in $L(\mathbb{R})$ assuming in V the existence of a limit of Woodin cardinals with a measurable above. However, we do not currently know of such a model of ZF over which forcing to add many Cohen subsets of a measurable cardinal preserves the measurability (see Question 5.7 in the next section). Nevertheless, we prove Theorem 4.4 in this generality with the optimism that such a model will be found.

Before going any further, let us recall the standard partition relation notation:

Definition 4.1. Given cardinals $\lambda, \kappa, d, \sigma$, the notation $\lambda \rightarrow (\kappa)_\sigma^d$ represents the statement that given any coloring of the size d subsets of λ using σ colors, there is a size κ subset of λ all of whose size d subsets are the same color.

We point out that Harrington's original forcing proof of $\text{HL}(d, k, \omega)$ can be lifted, with minor modifications, to obtain $\text{HL}(d, \sigma, \kappa)$ for κ measurable and $\sigma < \kappa$, provided, as in [19], we work in a model of ZFC in which κ remains measurable after forcing with $\text{Add}(\kappa, \lambda)$, where λ satisfies the partition relation $\lambda \rightarrow (\kappa^+)_2^{2d}$. An exposition of Harrington's original proof can be found in [3]. In order for this to

hold via methods of Woodin, one needs to assume the existence of a cardinal κ which is $\kappa + 2d$ -strong. Thus, in a somewhat straightforward manner, combining known results, one may arrive at a $\kappa + 2d$ -strong cardinal as an upper bound for the consistency strength of $\text{HL}(d, \sigma, \kappa)$ for κ measurable and $\sigma < \kappa$.

However, we are interested in the actual consistency strength of $\text{HL}(d, \sigma, \kappa)$ for κ measurable. The version of Harrington's forcing proof for perfect subtrees of ${}^{<\omega}2$ given by Todorćević in Theorem 8.15 of [9] uses the weaker partition relation $\lambda \rightarrow (\aleph_0)_2^d$, rather than the usual $\lambda \rightarrow (\aleph_0)_{2^d}^d$. If we could lift his argument to a measurable cardinal, this would bring the consistency strength down to a d -strong cardinal. One of the key lemmas in Todorćević's proof, though, relies strongly on the fact that \aleph_0 is the least infinite cardinal and could not be generalized to uncountable κ . However, using that κ is measurable, we found a way around this. We modified Harrington's argument and proved Lemma 4.3 which uses partition relations which are satisfied when κ is measurable (assuming $\delta < \omega$). This lemma aids in bringing the upper bound of the consistency strength of $\text{HL}(d, \sigma, \kappa)$ for κ measurable down to a $\kappa + d$ -strong cardinal in Theorem 4.7.

Definition 4.2. Temporarily let \mathbb{P} be the forcing to add λ Cohen subsets of κ . A collection $\mathcal{X} \subseteq \mathbb{P}$ is called *image homogenized* if for all $p_1, p_2 \in \mathcal{X}$ and $\xi, \alpha, \beta \in \text{Ord}$, if α is the ξ -th element of $\text{Dom}(p_1)$ and β is the ξ -th element of $\text{Dom}(p_2)$, then $p_1(\alpha) = p_2(\beta)$.

Suppose that we have parameterized conditions in \mathbb{P} according to $[\lambda]^\delta$. That is, for each $\vec{\alpha} \in [\lambda]^\delta$ we have some $p_{\vec{\alpha}}$. For each $\vec{\alpha} \in [\lambda]^\delta$, let

$$\langle \nu(\vec{\alpha}, \xi) : \xi < \text{ot}(\text{Dom}(p_{\vec{\alpha}})) \rangle$$

be the increasing enumeration of the elements of $\text{Dom}(p_{\vec{\alpha}})$.

Lemma 4.3. *Let $1 \leq \delta < \kappa \leq \lambda$ be ordinals with κ and λ infinite cardinals and κ strongly inaccessible. Assume that for all $\mu_1, \mu_2 < \kappa$,*

$$\kappa \rightarrow (\mu_1)_{\mu_2}^{\delta \cdot 2}.$$

For each $\vec{\alpha} \in [\kappa]^\delta$, let $p_{\vec{\alpha}}$ be a condition in the forcing to add λ many Cohen subsets of κ . Assume that the conditions $p_{\vec{\alpha}}$ for $\vec{\alpha} \in [\kappa]^\delta$ are image homogenized. Then for each $\gamma < \kappa$ there is a sequence

$$\langle H_i \subseteq \kappa : i < \delta \rangle$$

such that $(\forall i < \delta) \text{ot}(H_i) \geq \gamma$, $(\forall i < j < \delta)$ every element of H_i is less than every element of H_j , and the conditions $p_{\vec{\alpha}}$ for $\vec{\alpha} \in \prod_{i < \delta} H_i$ are pairwise compatible.

Proof. Consider a function ι from $\delta \cdot 2$ to $\delta \cdot 2$ such that $\iota \upharpoonright [0, \delta)$ is an increasing sequence of elements in $\delta \cdot 2$ and $\iota \upharpoonright [\delta, \delta \cdot 2)$ is an increasing sequence of elements in $\delta \cdot 2$. Given a set $A \in [\text{Ord}]^{\delta \cdot 2}$, let $\iota[A] := \langle \alpha_{\iota(i)} : i < \delta \cdot 2 \rangle$, where A enumerated in increasing order is $\langle \alpha_i : i < \delta \cdot 2 \rangle$. If $\vec{\alpha}, \vec{\beta} \in [\text{Ord}]^\delta$ are such that $(\exists A) \iota[A] = \langle \vec{\alpha}, \vec{\beta} \rangle$, then we say that $\vec{\alpha}$ is ι -related to $\vec{\beta}$. The idea is that ι codes the relative ordering of the elements of $\vec{\alpha}$ and $\vec{\beta}$. Given any $\vec{\alpha}, \vec{\beta} \in [\text{Ord}]^\delta$, there is some ι such that $\vec{\alpha}$ is ι -related to $\vec{\beta}$.

Call a function ι from $\delta \cdot 2$ to $\delta \cdot 2$ *acceptable* iff for all $i < \delta$,

$$\iota(i), \iota(\delta + i) \in \{2 \cdot i, 2 \cdot i + 1\}.$$

The idea is that when we have our final sequence $\langle H_i : i < \delta \rangle$, if $\vec{\alpha}, \vec{\beta} \in \prod_{i < \delta} H_i$, then $\vec{\alpha}$ will be ι -related to $\vec{\beta}$ for some ι that is acceptable. Also, as can be easily verified by the reader, given any acceptable ι , there are $\vec{\alpha}, \vec{\beta}, \vec{\sigma} \in \prod_{i < \delta} H_i$ such that $\vec{\alpha}$ is ι -related to $\vec{\beta}$, $\vec{\alpha}$ is ι -related to $\vec{\sigma}$, and $\vec{\beta}$ is ι -related to $\vec{\sigma}$.

For an acceptable $\iota : \delta \cdot 2 \rightarrow \delta \cdot 2$, let c_ι be a coloring of subsets of κ of size $\delta \cdot 2$ which encodes the following information. Given $A \in [\kappa]^{\delta \cdot 2}$, let $\vec{\alpha}$ and $\vec{\beta}$ be such that $\iota[A] = \langle \vec{\alpha}, \vec{\beta} \rangle$ (so $\vec{\alpha}$ is ι -related to $\vec{\beta}$). The value of $c_\iota(A)$ should encode

- 1) the relative ordering of the elements of $\text{Dom}(p_{\vec{\alpha}})$ and $\text{Dom}(p_{\vec{\beta}})$, and finally
- 2) whether or not $p_{\vec{\alpha}}$ and $p_{\vec{\beta}}$ are compatible.

This coloring c_ι can be seen to use strictly less than κ colors. In fact, since the conditions p_α , $\alpha \in [\kappa]^\delta$ are homogenized, their domains have the same size, say $\theta < \kappa$; so such a c_ι may be seen to use at most $|\theta \cdot 2| + 2$ colors. Define $c(A)$ to be the sequence of $c_\iota(A)$, where ι ranges over all acceptable ι , and notice that c uses less than κ many colors. Now fix $\gamma < \kappa$. By our assumption that

$$(\forall \mu_1, \mu_2 < \kappa) \kappa \rightarrow (\mu_1)_{\mu_2}^{\delta \cdot 2},$$

there is a set $H \subseteq \kappa$ of ordertype at least $\gamma \cdot \delta$ which is homogeneous for c , and hence, for each c_ι .

Partition H into δ pieces of size $\geq \gamma$ that are not interleaved (if P_1 and P_2 are two pieces, then if one element of P_1 is less than one element of P_2 , then all elements of P_1 are less than all elements of P_2). Let the partition be $\langle H_i : i < \delta \rangle$. We claim that for any two $\vec{\zeta}, \vec{\eta} \in \prod_{i < \delta} H_i$, $p_{\vec{\zeta}}$ and $p_{\vec{\eta}}$ are compatible. Suppose, towards a contradiction, that there are $\vec{\zeta}$ and $\vec{\eta}$ such that $p_{\vec{\zeta}}$ and $p_{\vec{\eta}}$ are incompatible. There is an ι such that $\vec{\zeta}$ is ι -related to $\vec{\eta}$ and ι is acceptable. Since $p_{\vec{\zeta}}$ and $p_{\vec{\eta}}$ are incompatible, fix $\sigma_1 \neq \sigma_2$ such that $\nu(\vec{\zeta}, \sigma_1) = \nu(\vec{\eta}, \sigma_2)$ but $p_{\vec{\zeta}}(\nu(\vec{\zeta}, \sigma_1)) \neq p_{\vec{\eta}}(\nu(\vec{\eta}, \sigma_2))$. The fact that it must be that $\sigma_1 \neq \sigma_2$ follows from the fact that the $p_{\vec{\alpha}}$ are image homogenized. Since whether or not $p_{\vec{\zeta}}$ and $p_{\vec{\eta}}$ are compatible was encoded into c_ι , any two $\vec{\alpha}, \vec{\beta} \in \prod_{i < \delta} H_i$ such that $\vec{\alpha}$ is ι -related to $\vec{\beta}$ will be such that $p_{\vec{\alpha}}$ and $p_{\vec{\beta}}$ are incompatible. Since ι is acceptable, fix $\vec{\alpha}, \vec{\beta}, \vec{\sigma} \in \prod_{i < \delta} H_i$ such that $\vec{\alpha}$ is ι -related to $\vec{\beta}$, $\vec{\alpha}$ is ι -related to $\vec{\sigma}$, and $\vec{\beta}$ is ι -related to $\vec{\sigma}$. By 1) of the definition of c_ι and the fact that the conditions are image homogenized, we have $\nu(\vec{\alpha}, \sigma_1) = \nu(\vec{\beta}, \sigma_2)$. Also, $\nu(\vec{\alpha}, \sigma_1) = \nu(\vec{\sigma}, \sigma_2)$ and $\nu(\vec{\beta}, \sigma_1) = \nu(\vec{\sigma}, \sigma_2)$. Thus, $\nu(\vec{\alpha}, \sigma_1) = \nu(\vec{\beta}, \sigma_1)$. Now we have $\nu(\vec{\beta}, \sigma_1) = \nu(\vec{\beta}, \sigma_2)$, which is impossible because ν is an increasing enumeration of the elements of $\text{Dom}(p_{\vec{\beta}})$ and $\sigma_1 \neq \sigma_2$. \square

Notice that 1) and 2) of the hypothesis of the theorem follow from $\lambda \rightarrow (\kappa)_{\kappa}^{\delta \cdot 2}$. The conclusion of this theorem clearly implies $\text{SDHL}(\delta, \sigma, \kappa)$, and hence also $\text{HL}(\delta, \sigma, \kappa)$ by Theorem 3.7. We point out that this proves the analogue for trees on κ of the asymmetric Dense Set Version, Theorem 8.15 in [9], for trees on ω .

Theorem 4.4. *Let κ be an infinite cardinal, $\delta < \kappa$ a non-zero ordinal, and $\lambda > \kappa$ a cardinal for which the following partition relations hold:*

- 1) $\lambda \rightarrow (\kappa)_{\kappa}^{\delta}$, and
- 2) $\kappa \rightarrow (\mu_1)_{\mu_2}^{\delta \cdot 2}$, for all pairs $\mu_1, \mu_2 < \kappa$.

Assume that κ is either ω or a measurable cardinal in the forcing extension to add λ many Cohen subsets of κ . Let $T_i \subseteq {}^{<\kappa}\kappa$ for $i < \delta$ be regular trees, and let $\sigma < \kappa$

be non-zero and

$$c : \bigotimes_{i < \delta} T_i \rightarrow \sigma.$$

Then there are $\zeta < \kappa$ and $\langle t_i \in T_i(\zeta) : i < \delta \rangle$, such that for all ζ' satisfying $\zeta \leq \zeta' < \kappa$, there are ζ'' satisfying $\zeta' \leq \zeta'' < \kappa$ and $\langle X_i \subseteq T_i(\zeta'') : i < \delta \rangle$ such that, for each $i < \delta$, X_i dominates $T_i(\zeta') \cap \text{Cone}(t_i)$, and

$$|c'' \bigotimes_{i < \delta} X_i| = 1.$$

Furthermore, we can ensure that if $c'' \bigotimes_{i < \delta} X_i = \{0\}$, then $(\forall i < \delta) t_i = \emptyset$.

In particular, $HL(\delta, \sigma, \kappa)$ holds.

Proof. Note that κ is strongly inaccessible. Let \mathbb{P} be the poset to add $\delta \times \lambda$ many Cohen subsets of κ presented in the following way: $p \in \mathbb{P}$ iff p is a size $< \kappa$ partial function from $\lambda \times \delta$ to ${}^{<\kappa}\kappa$ and for all $(\alpha, i) \in \text{Dom}(p)$, $p(\alpha, i) \in T_i$. For the ordering of \mathbb{P} , $q \leq p$ iff

$$\text{Dom}(q) \supseteq \text{Dom}(p) \text{ and } (\forall x \in \text{Dom}(p)) q(x) \supseteq p(x).$$

If each $T_i = {}^{<\kappa}\kappa$, then it is trivial to see that this forcing is equivalent to adding λ Cohen subsets of κ . If not, then as long as each T_i does not have any isolated branches or leaf nodes (which we assume), then because no path leaves tree T_i at a limit level $< \kappa$, one shows that the forcings are still equivalent. Without loss of generality, $(\forall p \in \mathbb{P})$ the size of $\text{Dom}(p) \cap (\lambda \times \{i\})$ does not depend on i . Similarly, we may assume $(\forall p \in \mathbb{P})$ the length of the sequence $p(x)$ does not depend on $x \in \text{Dom}(p)$.

Let \dot{G} be the canonical name for the generic object (as opposed to the generic filter). In particular, $1 \Vdash \dot{G} : \lambda \times \delta \rightarrow {}^{\kappa}\kappa$. Since $1 \Vdash (\kappa \text{ is a measurable cardinal})$, let \dot{U} be such that $1 \Vdash (\dot{U} \text{ is a } \kappa\text{-complete ultrafilter on } \kappa)$.

For each $\vec{\alpha} \in [\lambda]^\delta$, let $p_{\vec{\alpha}} \in \mathbb{P}$ be a condition and $\sigma_{\vec{\alpha}} < \sigma$ be a color such that if $\langle \alpha_i : i < \delta \rangle$ is the increasing enumeration of $\vec{\alpha}$, then

$$p_{\vec{\alpha}} \Vdash \{\zeta < \kappa : c(\langle \dot{G}(\alpha_i, i) \upharpoonright \zeta : i < \delta \rangle) = \check{\sigma}_{\vec{\alpha}}\} \in \dot{U}.$$

We may assume that if $\sigma_{\vec{\alpha}} = 0$, then $p_{\vec{\alpha}} = 1$. This is because if 1 does not force the color of \dot{U} many levels to be 0, then by the nature of the forcing relation and since \dot{U} is an ultrafilter, there is some condition p such that

$$p \Vdash \{\zeta < \kappa : c(\langle \dot{G}(\alpha_i, i) \upharpoonright \zeta : i < \delta \rangle) \neq 0\} \in \dot{U}.$$

This will establish the last sentence of the statement of the theorem. Then, since $\sigma < \kappa$ and \dot{U} is κ -complete (in the extension), there is $p' \leq p$ and $\sigma \neq 0$ such that

$$p' \Vdash \{\zeta < \kappa : c(\langle \dot{G}(\alpha_i, i) \upharpoonright \zeta : i < \delta \rangle) = \sigma\} \in \dot{U}.$$

Then set $p_{\vec{\alpha}} = p'$ and $\sigma_{\vec{\alpha}} = \sigma$ and we are done. We may also assume, by possibly making the conditions $p_{\vec{\alpha}}$ stronger, that for each $\vec{\alpha} \in [\lambda]^\delta$ and $i < \delta$, that

$$(\alpha_i, i) \in \text{Dom}(p_{\vec{\alpha}}).$$

There is a coloring $\tilde{c} : [\lambda]^\delta \rightarrow \kappa$ such that after we apply the partition relation $\lambda \rightarrow (\kappa)_{\kappa}^\delta$, we get a set $H \in [\lambda]^\kappa$ such that the following are satisfied:

- 1) the sequence $\langle (\xi, p_{\vec{\alpha}}(\nu(\vec{\alpha}, \xi))) : \xi < \text{ot}(\text{Dom}(p_{\vec{\alpha}})) \rangle$ does not depend on $\vec{\alpha} \in [H]^\delta$ and therefore the set of $p_{\vec{\alpha}}$ for $\vec{\alpha} \in [H]^\delta$ are image homogenized,
- 2) the value $\sigma_{\vec{\alpha}}$ does not depend on $\vec{\alpha} \in [H]^\delta$, and

- 3) for a fixed $i < \delta$, the sequence $\langle p_{\vec{\alpha}}(\alpha_i, i) \in T_i : i < \delta \rangle$ does not depend on $\vec{\alpha} \in [H]^\delta$.

Such a coloring \tilde{c} may be defined as follows. For $\vec{\alpha} \in [\lambda]^\delta$, define $\tilde{c}(\vec{\alpha})$ to be the triple sequence consisting of the following three sequences: the sequence from 1); the value $\sigma_{\vec{\alpha}}$ that $p_{\vec{\alpha}}$ forces $c(\langle \dot{G}(\alpha_i, i) \upharpoonright \zeta : i < \delta \rangle)$ to equal for \mathcal{U} many levels ζ ; and the sequence $\langle \xi_i : i < \delta \rangle$, where ξ_i is the ordinal such that α_i appears as the ξ_i -th member of $\text{ot}(\text{Dom}(p_{\vec{\alpha}}))$.

For each $i < \delta$, let $t_i \in {}^{<\kappa}\kappa$ be the unique value of the $p_{\vec{\alpha}}(i, \alpha_i)$'s for $\vec{\alpha} = \langle \alpha_i : i < \delta \rangle \in [H]^\delta$. Let ζ be the common length of each t_i . Let $\sigma' < \sigma$ be the unique value of $\sigma_{\vec{\alpha}}$ for $\vec{\alpha} \in [H]^\delta$. Note that if $\sigma' = 0$, then each $p_{\vec{\alpha}}$ for $\vec{\alpha} \in [H]^\delta$ equals 1, so $(\forall i < \delta) t_i = \emptyset$.

Now pick an arbitrary ζ' satisfying $\zeta \leq \zeta' < \kappa$. Let $\gamma < \kappa$ be a cardinal such that for each $i < \delta$, the cardinality of the set $T(\zeta') \cap \text{Cone}(t_i)$ is $\leq \gamma$. Now apply Lemma 4.3 to get a sequence $\langle H_i \subseteq H : i < \delta \rangle$ such that $(\forall i < \delta) \text{ot}(H_i) \geq \gamma$, $(\forall i < j < \delta)$ every element of H_i is less than every element of H_j , and the conditions $p_{\vec{\alpha}}$ for $\vec{\alpha} \in \prod_{i < \delta} H_i$ are pairwise compatible. To apply this lemma, we needed hypothesis 2) of this theorem (to hold in the ground model). We could instead apply Lemma 4.3 in the forcing extension as long as 2) holds in the extension, and the sequence $\langle H_i : i < \delta \rangle$ would be in the ground model because the forcing is $<\kappa$ -closed.

For each $i < \delta$, let $S_i := T_i(\zeta') \cap \text{Cone}(t_i)$, the set of successors of t_i in T_i . For each $i < \delta$ and $t \in S_i$, pick $\alpha_{i,t} \in H_i$ so that every element of S_i is mapped to a unique element of H_i . This is possible because the cardinality of S_i is $\leq \gamma$ and γ is \leq the ordertype of H_i . For $i < \delta$, let $A_i := \{\alpha_{i,t} : t \in S_i\} \subseteq H_i$. Let $\mathcal{X} \subseteq \prod_{i < \delta} H_i$ be the set $\mathcal{X} := \prod_{i < \delta} A_i$. The conditions $p_{\vec{\alpha}}$ for $\vec{\alpha} \in \mathcal{X}$ are pairwise compatible.

Let $p \in \mathbb{P}$ be a condition which extends $\bigcup \{p_{\vec{\alpha}} : \vec{\alpha} \in \mathcal{X}\}$ and for all $i < \delta$ and $t \in S_i$, $p(\alpha_{i,t}, i) = t$. To see that there exists such a p , first note that $|\mathcal{X}| < \kappa$ and the conditions $p_{\vec{\alpha}} \in \mathcal{X}$ are pairwise compatible, therefore by the nature of \mathbb{P} , $\bigcup_{\vec{\alpha} \in \mathcal{X}} p_{\vec{\alpha}}$ is an element of \mathbb{P} . Second, since $(\forall \vec{\alpha} \in \mathcal{X})(\forall i < \delta) p_{\vec{\alpha}}(\alpha_i, i) = t_i$, we may define p so that $p(i, \alpha_{i,t}) = t$ for all $i < \delta$ and $t \in S_i$ and this will not clash with the $p_{\vec{\alpha}}$.

Since $|\mathcal{X}| < \kappa$, $1 \Vdash (\mathcal{U} \text{ is a } \tilde{\kappa}\text{-complete ultrafilter})$, and by the hypothesis on each pair $(p_{\vec{\alpha}}, \sigma_{\vec{\alpha}})$, we have that $p \Vdash$ there are arbitrarily large levels $\zeta'' < \tilde{\kappa}$ such that for each $\vec{\alpha} \in \mathcal{X}$, we have

$$\check{c}(\langle \dot{G}(\alpha_i, i) \upharpoonright \zeta'' : i < \delta \rangle) = \sigma'.$$

We may extend p to a condition $p' \leq p$ as well as get a level $\zeta'' \in [\zeta', \kappa)$ such that the following are satisfied:

- 1) $p' \Vdash$ for each $\vec{\alpha} = \langle \alpha_i : i < \delta \rangle \in \mathcal{X}$, we have

$$\check{c}(\langle \dot{G}(\alpha_i, i) \upharpoonright \zeta'' : i < \delta \rangle) = \sigma'.$$

- 2) for each $i < \delta$ and each $t \in S_i$, there is a unique $\tilde{t} \in T_i(\zeta'')$ such that $p' \Vdash \dot{G}(\tilde{\alpha}_{i,t}, \tilde{i}) \supseteq \tilde{t}$.

For each $i < \delta$, let

$$X_i := \{\tilde{t} : t \in S_i\}.$$

We have that each X_i dominates S_i and

$$c'' \bigotimes_{i < \delta} X_i = \{\sigma'\}$$

(because the coloring c is in the ground model, so it is absolute). \square

The following theorem of Erdős and Rado provides many examples of cardinals λ which satisfy the hypothesis in Theorem 4.4, when δ is finite.

Theorem 4.5 (Erdős-Rado, [8]). *For $r \geq 0$ finite and κ an infinite cardinal, $\beth_r(\kappa)^+ \rightarrow (\kappa^+)_\kappa^{r+1}$.*

In particular, if GCH holds, then for finite $d \geq 2$, $\kappa^{+d} \rightarrow (\kappa^+)_\kappa^d$.

Definition 4.6 ([14]). Given a cardinal κ and an ordinal d , κ is $\kappa + d$ -strong if there is an elementary embedding $j : V \rightarrow M$ with critical point κ such that $V_{\kappa+d} = M_{\kappa+d}$.

Theorem 4.7. *Let $d \geq 1$ be any finite integer and suppose κ is a $\kappa + d$ -strong cardinal in a model V of ZFC satisfying GCH. Then there is a forcing extension in which κ remains measurable and $\text{HL}(d, \sigma, \kappa)$ holds, for all $\sigma < \kappa$.*

Proof. Let $\langle \kappa_\alpha : \alpha < \kappa \rangle$ enumerate all the strongly inaccessible cardinals in V below κ . Let \mathbb{P}_κ denote the κ -length reverse Easton support iteration of $\text{Add}(\kappa_\alpha, \kappa_\alpha^{+d})$, and let G_κ be \mathbb{P}_κ -generic over V . Then in $V[G_\kappa]$, κ is still measurable by a standard lifting of the embedding, and GCH holds at and above κ since $|\mathbb{P}_\kappa| = \kappa$. Thus, in $V[G_\kappa]$, the partition relation $\kappa^{+d} \rightarrow (\kappa)_\kappa^d$ holds. Since κ is measurable in $V[G_\kappa]$, the partition relations $\kappa \rightarrow (\mu_1)_{\mu_2}^{d-2}$ holds for all pairs $\mu_1, \mu_2 < \kappa$. Let \mathbb{Q} denote $\text{Add}(\kappa, \kappa^{+d})$ in $V[G_\kappa]$, and let H be \mathbb{Q} -generic over $V[G_\kappa]$. By an unpublished result of Woodin (see for instance [10] for a proof) κ remains measurable in $V[G_\kappa][H]$, since κ is $\kappa + d$ -strong in V . Thus, the hypotheses of Theorem 4.4 are satisfied in $V[G_\kappa]$, and therefore $\text{HL}(d, \sigma, \kappa)$ holds in $V[G_\kappa]$. \square

We conclude this section by pointing out that, in place of $\text{Add}(\kappa_\alpha, \kappa_\alpha^{+d})$, one could use a reverse Easton iteration of κ_α^{+d} -products with $< \kappa_\alpha$ -support of κ_α -Sacks forcing, as in [10], to achieve a model with κ measurable and $\text{HL}(d, \sigma, \kappa)$ holding. However, the homogeneity argument in the body of Theorem 4.4 would use κ^+ colors, and so we would need to start with a cardinal κ which is $(\kappa + d + 1)$ -strong in the ground model to get an analogue of Theorem 4.7. It should be noted though that for trees of height ω , Harrington's original forcing proof can also be modified to use Sacks forcing; the larger λ needed to accommodate the homogeneity argument would need to satisfy $\lambda \rightarrow (\omega)_{2^\omega}^{2^d}$.

5. CLOSING COMMENTS AND OPEN PROBLEMS

The following conjectures and questions are motivated by the results in the previous sections and their comparisons with results in [19], [11] and [6]. We have found upper bounds for the consistency strengths of $\text{HL}(d, \sigma, \kappa)$ for κ measurable and are interested in the exact consistency strength.

Conjecture 5.1. For finite $d \geq 2$, κ measurable and $\sigma < \kappa$, the consistency strength of $\text{HL}(d, \sigma, \kappa)$ is a $\kappa + d$ -strong cardinal.

If the conjecture turns out to be true, then there must be a positive answer to the next question.

Question 5.2. Given $d \geq 1$, is there a model of ZFC in which there is measurable cardinal κ such that $\text{HL}(d, \sigma, \kappa)$ holds for all $\sigma < \kappa$, but $\text{HL}(d + 1, \sigma, \kappa)$ fails for some $2 \leq \sigma < \kappa$?

In Section 8 of [6], it is mentioned that a model satisfying the hypotheses of Theorem 2.5 in [6] can be constructed, assuming the existence of a measurable cardinal κ such that $o(\kappa) = \kappa^{+2m+2}$. We conjecture that the form of Halpern-Läuchli in [19] and [6] is strictly stronger than the form $\text{HL}(d, \sigma, \kappa)$.

Conjecture 5.3. Let $d \geq 2$ be a finite number. The consistency strength of $\text{HL}(d, \sigma, \kappa)$ for κ measurable is strictly less than the consistency strength of Theorem 2.5 in [6] for coloring d -sized antichains.

Further, Džamonja, Larson, and Mitchell point out that Theorem 2.5 in [6] is not a consequence of any large cardinal assumption. This follows from results of Hajnal and Komjáth in [11], a consequence of which is that there is a forcing of size \aleph_1 after which there is a coloring of the pairsets on ${}^{<\kappa}2$ for which there is no strong subtree homogeneous for the coloring. However, that theorem of Hajnal and Komjáth does not seem to immediately provide a counterexample to $\text{HL}(d, 2, \kappa)$, and so we ask the following.

Question 5.4. For $d \geq 2$, is there a large cardinal assumption on κ which implies $\text{HL}(d, 2, \kappa)$ holds?

In fact, the following is open:

Question 5.5. Is $\text{HL}(2, 2, \kappa)$ for some $\kappa > \omega$ a theorem of ZFC?

In Section 3 we showed that all the variants considered in this paper are equivalent as long as κ is weakly compact, and showed that $\text{HL}(1, k, \kappa)$ holds when κ is infinite and k is finite. In Section 8 of [6], an argument is provided showing that any cardinal κ satisfying their Theorem 2.5 for $m \geq 2$ must be weakly compact. However, that argument does not seem to apply to our situation and so we ask the following.

Question 5.6. For $d \geq 2$, is the statement “ $\kappa \geq \aleph_1$ and $\text{HL}(d, 2, \kappa)$ holds” equiconsistent with some large cardinal?

Lastly, we proved Theorem 4.4 for the general case of infinitely many trees with the optimism that a model of ZF satisfying the conditions can be found, and hence ask the following.

Question 5.7. Is there a model of ZF satisfying the partition relations stated in the hypotheses of Theorem 4.4 for some infinite $\delta < \kappa$ such that after forcing with $\text{Add}(\kappa, \lambda)$, κ is measurable?

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