DISJOINT BOREL FUNCTIONS

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Abstract. For each $a \in {}^\omega\omega$, we define a Baire class one function $f_a : {}^\omega\omega \to {}^\omega\omega$ which encodes a in a certain sense. We show that for each Borel $g : {}^\omega\omega \to {}^\omega\omega$, $f_a \cap g = \emptyset$ implies $a \in \Delta^1_1(c)$ where c is any code for g. We generalize this theorem for g in a larger pointclass Γ. Specifically, when $\Gamma = \boldsymbol{\Delta}^1_2$, $a \in L[c]$. Also for all $n \in \omega$, when $\Gamma = \boldsymbol{\Delta}^1_{3+n}$, $a \in \mathcal{M}_{1+n}(c)$.

1. Introduction

Definition 1.1. A challenge-response relation (*c.r.-relation*) is a triple $\langle R_-, R_+, R\rangle$ such that $R \subseteq R_- \times R_+$. The set R_- is the set of *challenges*, and R_+ is the set of *responses*. When cRr, we say that r *meets* c.

Definition 1.2. A backwards generalized Galois-Tukey connection (*morphism*) from $\mathcal{A} = \langle A_-, A_+, A\rangle$ to $\mathcal{B} = \langle B_-, B_+, B\rangle$ is a pair $\langle \phi_-, \phi_+\rangle$ of functions $\phi_- : B_- \to A_-$ and $\phi_+ : A_+ \to B_+$ such that

$$(\forall c \in B_-)(\forall r \in A_+)\, \phi_-(c)\, A\, r \Rightarrow c\, B\, \phi_+(r).$$

When there is a morphism from $\mathcal{A}$ to $\mathcal{B}$, let us say that $\mathcal{A}$ is *above* $\mathcal{B}$ and $\mathcal{B}$ is *below* $\mathcal{A}$.

Definition 1.3. The *norm* of a c.r.-relation $\mathcal{R} = \langle R_-, R_+, R\rangle$ is

$$||\mathcal{R}|| := \min\{|S| : S \subseteq R_+ \text{ and } (\forall c \in R_-)(\exists r \in S)\, c\, R\, r\}.$$

If there is a morphism from $\mathcal{A}$ to $\mathcal{B}$, then $||\mathcal{A}|| \geq ||\mathcal{B}||$. Challenge-response relations and morphisms between them were introduced by Vojtas as a way to abstract features of the study of cardinal charcteristics of the continuum. For more on c.r.-relations, see [2] and [6].

Temporarily fix a pointclass Γ. Let $\mathcal{F}_\Gamma$ be the set of functions from ${}^\omega\omega$ to ${}^\omega\omega$ in Γ. Let D be the binary relation of disjointness of functions from ${}^\omega\omega$ to ${}^\omega\omega$. That is, given two functions $f, g : {}^\omega\omega \to {}^\omega\omega$, let

$$f D g :\Leftrightarrow f \cap g = \emptyset \Leftrightarrow (\forall x \in {}^\omega\omega)\, f(x) \neq g(x).$$

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Let $\mathcal{D}_\Gamma$ be the c.r.-relation

$$\mathcal{D}_\Gamma := \langle \mathcal{F}_\Gamma, \mathcal{F}_\Gamma, D \rangle.$$

In this paper we will be interested in the c.r.-relation $\mathcal{D}_\Gamma$ for various pointclasses Γ.

For example, we will be interested in computing $||\mathcal{D}_{\mathbf{\Delta}^1_1}||$, which is the smallest size of a family of Borel functions from ${}^\omega\omega$ to ${}^\omega\omega$ such that each Borel function from ${}^\omega\omega$ to ${}^\omega\omega$ is disjoint from some member of the family. We will show that $||\mathcal{D}_{\mathbf{\Delta}^1_1}|| = 2^\omega$ by showing that $\mathcal{D}_{\mathbf{\Delta}^1_1}$ is above a c.r.-relation whose norm is 2^ω. Specifically, we will show that $\mathcal{D}_{\mathbf{\Delta}^1_1}$ is above $\langle {}^\omega\omega, {}^\omega\omega, \le_{\Delta^1_1} \rangle$, where $a \le_{\Delta^1_1} b$ iff $a \in {}^\omega\omega$ is definable by a Δ^1_1 formula using $b \in {}^\omega\omega$ as a parameter. To define the ϕ_- part of the morphism, for each $a \in {}^\omega\omega$ we will define a Baire class one funciton $f_a : {}^\omega\omega \to {}^\omega\omega$ (and we will have $\phi_-(a) = f_a$). The ϕ_+ part of the morphism will simply map each function from ${}^\omega\omega$ to ${}^\omega\omega$ in Γ to any code for that function. The fact that $\langle \phi_-, \phi_+ \rangle$ is a morphism is the following statement: for each $a \in {}^\omega\omega$ and Borel function $g : {}^\omega\omega \to {}^\omega\omega$,

$$f_a \cap g = \emptyset \Rightarrow a \le_{\Delta^1_1} \text{any code for } g.$$

We will prove that there is a morphism from $\mathcal{D}_{\mathbf{\Delta}^1_1}$ to $\langle {}^\omega\omega, {}^\omega\omega, \le_{\Delta^1_1} \rangle$ by proving a general theorem (Theorem 5.3) which provides a sufficient condition for when there exists a morphism from an arbitrary $\mathcal{D}_\Gamma$ to an arbitrary $\langle {}^\omega\omega, {}^\omega\omega, \prec \rangle$, where $\prec$ is an ordering on ${}^\omega\omega$. Just like the case with $\mathcal{D}_{\mathbf{\Delta}^1_1}$, we will use the functions f_a for the ϕ_- map, and the ϕ_+ map will be "take any code for". Thus, if the appropriate relationship holds between Γ and $\prec$, then we will have that for each $a \in {}^\omega\omega$ and each $g : {}^\omega\omega \to {}^\omega\omega$ in Γ,

$$f_a \cap g = \emptyset \Rightarrow a \prec \text{any code for } g.$$

We will get that there exists a morphism from $\mathcal{D}_{\mathbf{\Delta}^1_2}$ to $\langle {}^\omega\omega, {}^\omega\omega, \le_L \rangle$, where $a \le_L b$ iff $a \in L[b]$. The analogous result for larger Γ uses large cardinals. We will have that as long as $\mathcal{M}_1(b)$ (the canonical inner model containing 1 Woodin cardinal and containing $b \in {}^\omega\omega$) exists for all $b \in {}^\omega\omega$, then there is a morphism from $\mathcal{D}_{\mathbf{\Delta}^1_3}$ to $\langle {}^\omega\omega, {}^\omega\omega, \le_{\mathcal{M}_1} \rangle$, where $a \le_{\mathcal{M}_1} b$ iff $a \in \mathcal{M}_1(b)$. Next, as long as $\mathcal{M}_2(b)$ exists for all $b \in {}^\omega\omega$, there is a morphism from $\mathcal{D}_{\mathbf{\Delta}^1_4}$ to $\langle {}^\omega\omega, {}^\omega\omega, \le_{\mathcal{M}_2} \rangle$. The pattern continues like this through the projective hierarchy.

In this paper, we are considering functions from ${}^\omega\omega$ to ${}^\omega\omega$ in a pointclass Γ. We could have instead considered functions in Γ from an arbitrary uncountable Polish space X to an arbitrary Polish space Y, and our results would not change much. The appropriate encoding function $f''_a : X \to Y$ could be defined by first defining $f'_a : {}^\omega 2 \to {}^\omega\omega$

in a way similar to f_a and then using an injection of ${}^{\omega}2$ into X and a surjection of ${}^{\omega}\omega$ onto Y. We trust that the interested reader can work through the details without trouble.

2. Related Results

Before considering $\mathcal{D}_\Gamma$ for various Γ, we will consider related c.r.-relations. First, consider the everywhere domination ordering of functions from ${}^{\omega}\omega$ to ω. That is, given $f, g : {}^{\omega}\omega \to \omega$, we write $f \leq g$ iff

$$(\forall x \in {}^{\omega}\omega)\, f(x) \leq g(x).$$

Given any pointclass Γ, let $\mathcal{E}_\Gamma$ be the c.r.-relation whose challenges and responses are Γ functions from ${}^{\omega}\omega$ to ω, and g meets f iff $f \leq g$.

Next, consider the pointwise eventual domination ordering of functions from ${}^{\omega}\omega$ to ${}^{\omega}\omega$. That is, given $f, g : {}^{\omega}\omega \to {}^{\omega}\omega$, we write $f \leq^* g$ iff

$$(\forall x \in {}^{\omega}\omega)\{n \in \omega : f(x)(n) > g(x)(n)\} \text{ is finite.}$$

Given any pointclass Γ, let $\mathcal{R}_\Gamma$ be the c.r.-relation whose challenges and responses are Γ functions from ${}^{\omega}\omega$ to ${}^{\omega}\omega$, and g meets f iff $f \leq^* g$.

It is not difficult to see that for any reasonably closed pointclass Γ, there is a morphism from $\mathcal{E}_\Gamma$ to $\mathcal{R}_\Gamma$ and there is a morphism from $\mathcal{R}_\Gamma$ to $\mathcal{D}_\Gamma$. The relation $\mathcal{E}_\Gamma$ for a fixed Γ is relatively high up in the hierarchy of c.r.-relations, as we will soon see.

Given a sequence $a \in {}^{\omega}\omega$, let $[[a]] := \{a \restriction l : l \in \omega\}$. Given a tree $T \subseteq {}^{<\omega}\omega$, let $\mathrm{Exit}(T)$ be the (Baire class one) function

$$\mathrm{Exit}(T)(x) := \min\{l : x \restriction l \notin T\}.$$

The following result shows a way of constructing a morphism from $\mathcal{E}_\Gamma$ to another relation in a way which does not depend on Γ:

Theorem 2.1. *Fix $a \in {}^{\omega}\omega$. If M is an ω-model ZF such that some $g : ({}^{\omega}\omega)^M \to \omega$ in M satisfies*

$$(\forall x \in ({}^{\omega}\omega)^M)\, \mathrm{Exit}([[a]])(x) \leq g(x),$$

then a is Δ^1_1 definable in M using g as a predicate.

Proof. Fix M and g satisfying the hypothesis of the theorem. Let $B \subseteq {}^{<\omega}\omega$ be the set

$$\{t \in {}^{<\omega}\omega : g(x) \geq |t| \text{ for all } x \sqsupseteq t \text{ in } M\}.$$

Note that B is defined (in M) by a Π^1_1 formula that uses g as a predicate. That is, B is Π^1_1 in g. We claim there is some $l \in \omega$ satisfying $(\forall l' \geq l)\, a \restriction l' \notin B$. If not, the poset of elements of B ordered by extension would be ill-founded, and therefore would be ill-founded in

M, so there would exist $x \in ({}^\omega\omega)^M$ satisfying $(\exists^\infty l' \in \omega)\, g(x) \geq l'$, which is impossible. Now, fix such an l.

We claim that for each $l' \geq l$, $a(l')$ is the unique n satisfying $(a \restriction l')^\frown n \notin B$. Indeed, since $\mathrm{Exit}([[a]]) \leq g$, for each $l' \geq l$ we have

$$(\forall n \in \omega)\, a(l') \neq n \Rightarrow (a \restriction l')^\frown n \in B.$$

The other direction is given by the property we arranged l to have. Thus, we have the following definition (in M) for a:

$$a(l') = \begin{cases} a(l') & \text{if } l' < l, \\ n & \text{if } l' \geq l \text{ and } (\forall n' \neq n)(\forall x \sqsupseteq (a \restriction l')^\frown n' \text{ in } M)\, g(x) \geq l' + 1. \end{cases}$$

Since $\langle a(l') : l' < l \rangle$ can be coded by a single number, we have a Π^1_1 definition (in M) for a which uses g as a predicate. We also have a Σ^1_1 variant:

$$a(l') = \begin{cases} a(l') & \text{if } l' < l, \\ n & \text{if } l' \geq l \text{ and } (\exists x \sqsupseteq (a \restriction l')^\frown n \text{ in } M)\, g(x) < l' + 1. \end{cases}$$

Thus, a is Δ^1_1 definable in M using g as a predicate. □

Let use write "*All*" to refer to the pointclass of all pointsets.

Corollary 2.2. *There is a morphism from $\mathcal{E}_{All}$ to $\langle {}^\omega\omega, {}^{({}^\omega\omega)}\omega, \leq_{\Delta^1_1} \rangle$.*

Proof. Fix $a \in {}^\omega\omega$. Let $f_a := \mathrm{Exit}([[a]])$. By the above theorem taking $M = V$, if $g : {}^\omega\omega \to \omega$ satisfies $f_a \leq g$, then a is Δ^1_1 definable using g as a predicate. □

Corollary 2.3. *There is a morphism from $\mathcal{E}_{\boldsymbol{\Delta}^1_1}$ to $\langle {}^\omega\omega, {}^\omega\omega, \leq_{\Delta^1_1} \rangle$.*

Proof. Fix $a \in {}^\omega\omega$. Let $f_a := \mathrm{Exit}([[a]])$. Let $g : {}^\omega\omega \to \omega$ be Borel and let c be a code for g. If we can show that a is in every ω-model which contains c, we will have that $a \leq_{\Delta^1_1} c$. Let M be an arbitrary ω-model which contains c. Letting $\tilde{g}$ be the function in M coded by c, we have that $\tilde{g} = M \cap g$. Hence, in M we have $f_a \leq \tilde{g}$, so the theorem above tells us that $a \in M$. □

Corollary 2.3 will be improved by our result that there is a morphism from $\mathcal{D}_{\boldsymbol{\Delta}^1_1}$ to $\langle {}^\omega\omega, {}^\omega\omega, \leq_{\Delta^1_1} \rangle$. The generalizations of Corollary 2.3 to larger pointclasses Γ are also improved by our main result (Theorem 5.3) about morphisms from $\mathcal{D}_\Gamma$ to orderings $\langle {}^\omega\omega, {}^\omega\omega, \prec \rangle$. On the other hand, we do not have an analogue of Corollary 2.2 with $\mathcal{D}_{All}$; here we see a qualitative difference between $\mathcal{E}_{All}$ and $\mathcal{D}_{All}$.

Another difference between $\mathcal{E}_{All}$ and $\mathcal{D}_{All}$ is the ability to encode not just an $a \in {}^\omega\omega$ but an $A \subseteq {}^\omega\omega$:

Proposition 2.4. *Fix a set X. Fix $A \subseteq X$. There exists a function $f_A : {}^\omega X \to \omega$ such that whenever M is a transitive model of ZF with $X \in M$ and M contains some $g : ({}^\omega X)^M \to \omega$ satisfying*

$$(\forall x \in ({}^\omega X)^M)\, f_A(x) \le g(x),$$

then $A \in M$. Moreover, there is some $t \in {}^{<\omega}X$ satisfying

$$A = \{z \in X : g(x) \ge |t| + 1 \text{ for all } x \sqsupseteq t^\frown z \text{ in } M\}.$$

Proof. It suffices to show the second claim. Let $f_A : {}^\omega X \to \omega$ be the function

$$f_A(x) := \begin{cases} 0 & \text{if } (\forall l \in \omega)\, x(l) \notin A, \\ l+1 & \text{if } x(l) \in A \text{ and } (\forall l' < l)\, x(l') \notin A. \end{cases}$$

Define

$$B := \{t \in {}^{<\omega}X : g(x) \ge |t| \text{ for all } x \sqsupseteq t \text{ in } M\}.$$

We must find a $t \in {}^{<\omega}X$ satisfying

$$A = \{z \in X : t^\frown z \in B\},$$

and we will be done. By the hypothesis on g and the definition of f_A, for each $z \in X$, $z \in A$ implies $\langle z \rangle \in B$. If conversely for each $z \in X$, $\langle z \rangle \in B$ implies $z \in A$, then we have

$$A = \{z \in X : \langle z \rangle \in B\},$$

and we are done by defining $t := \emptyset$. If not, then fix some $x_0 \in X$ satisfying $\langle x_0 \rangle \in B$ but $x_0 \notin A$.

Again by the hypothesis on g and the definition of f_A, for each $z \in X$, $z \in A$ implies $\langle x_0, z \rangle \in B$. Here it is important that $x_0 \notin A$. Again, if the converse holds that $\langle x_0, z \rangle \in B$ implies $z \in A$, then

$$A = \{z \in X : \langle x_0, z \rangle \in B\},$$

and we are done by defining $t := \langle x_0 \rangle$. If not, we may fix $x_1 \in X$ satisfying $\langle x_0, x_1 \rangle \in B$ but $x_1 \notin A$. We may continue like this, but we claim that the procedure terminates in a finite number of steps.

Assume, towards a contradiction, that it does not terminate. The sequence

$$x := \langle x_0, x_1, ... \rangle$$

we have constructed has all its initial segments in B. However, x need not be in M. We handle this situation as follows: let T be the set of those elements of B all of whose initial segments are also in B. The tree T is ill-founded because x is a path through it. Since being ill-founded is absolute, T has some path x' in M. We now have $(\forall l \in \omega)\, g(x') \ge l$, which is impossible. □

We immediately have the following:

Corollary 2.5. *For each $A \subseteq {}^\omega\omega$, there is a function $f_A : {}^\omega\omega \to \omega$ such that whenever $g : {}^\omega\omega \to \omega$ is any function which satisfies $f \leq g$, then A is $\mathbf{\Delta}^1_1$ in a predicate for g. Thus, there is a morphism from $\mathcal{E}_{All}$ to $\langle \mathcal{P}({}^\omega\omega), \mathcal{P}({}^\omega\omega), \leq_{\mathbf{\Delta}^1_1} \rangle$.*

Proof. Use the above theorem with $X = {}^\omega\omega$ and $M = V$. □

Now, a morphism from $\mathcal{D}_{All}$ to $\langle \mathcal{P}({}^\omega\omega), \mathcal{P}({}^\omega\omega), \prec \rangle$, where $\prec$ is any ordering such that $(\forall B \in \mathcal{P}({}^\omega\omega))\, |\{A : A \prec B\}| \leq 2^\omega$, will imply that $||\mathcal{D}_{All}|| = 2^{2^\omega}$. However, it is consistent that $||\mathcal{D}_{All}|| < 2^{2^\omega}$ so there can be no such morphism. In fact, it is consistent that $||\mathcal{R}_{All}|| < 2^{2^\omega}$. This contrasts with the fact that $||\mathcal{E}_{All}|| = 2^{2^\omega}$.

To get a model of $||\mathcal{R}_{All}|| < 2^{2^\omega}$, it suffices to get a model in which $\mathfrak{b} = \mathfrak{c}$ (so that there is a scale in $\langle {}^\omega\omega, \leq^* \rangle$ of length $\mathfrak{c}$) and the cofinality $\mathrm{cf}\langle {}^\mathfrak{c}\mathfrak{c}, \leq \rangle$ of all functions from $\mathfrak{c}$ to $\mathfrak{c}$ ordered by everywhere domination is $< 2^\mathfrak{c}$. By $\langle {}^\lambda\lambda, \leq^* \rangle$ we mean the set of functioms from λ to λ ordered by domination mod $< \lambda$. By $\mathfrak{b}$ we mean the *bounding number*, and $\mathfrak{c} = 2^\omega$. To get the required model, we first force so that 1) $\mathfrak{t} = \mathfrak{c}$ (where $\mathfrak{t}$ is the *tower number*), 2) $\mathfrak{c}$ is regular, 3) $\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$, and 4) $\mathfrak{c}^+ < 2^\mathfrak{c}$. Then, we force to add $\mathfrak{c}^{++}$ Cohen subsets of $\mathfrak{c}$. This preserves 1)-4). Finally, we force by the generalization of Hechler forcing in [3] to cofinally embed $\langle \mathfrak{c}^+, \leq \rangle$ into the poset of functions from $\mathfrak{c}$ to $\mathfrak{c}$ ordered by eventual mod $< \mathfrak{c}$ domination ($\leq^*$). A simple observation shows that $\mathrm{cf}\langle {}^\mathfrak{c}\mathfrak{c}, \leq \rangle = \mathrm{cf}\langle {}^\mathfrak{c}\mathfrak{c}, \leq^* \rangle$, and we are done.

For the last result of this section, let $\mathcal{V}_{\mathbf{\Delta}^1_1}$ be the c.r.-relation whose challenges and responses are Borel functions from ${}^\omega\omega \times {}^\omega\omega$ to ω, and g meets f iff $(\forall x \in {}^\omega\omega)(\exists y \in {}^\omega\omega)\, f(x,y) = g(x,y)$. By Theorem 5.3 we will have that $||\mathcal{D}_{\mathbf{\Delta}^1_1}|| = 2^\omega$. It is natural to ask whether $||\mathcal{V}_{\mathbf{\Delta}^1_1}|| = 2^\omega$. The answer is no for the following reason: fix an $\alpha < \omega_1$. Using the fact that there is a universal $\mathbf{\Sigma}^0_\alpha$ set, we can build a function $g_\alpha : {}^\omega\omega \times {}^\omega\omega \to \omega$ whose graph is $\mathbf{\Sigma}^0_{\alpha+1}$ such that if $f : {}^\omega\omega \times {}^\omega\omega \to \omega$ is a function whose graph is $\mathbf{\Sigma}^0_\alpha$, then g_α meets f. Hence, $||\mathcal{V}_{\mathbf{\Delta}^1_1}|| = \omega_1$

3. The Encoding Function

In this section we will define the function $f_a : {}^\omega\omega \to {}^\omega\omega$ which encodes $a \in {}^\omega\omega$ to be used in Thorem 5.3.

Definition 3.1 (The Encoding Function f_a)**.** Fix $a \in {}^\omega\omega$. Pick some $A \subseteq \omega$ such that $A =_T a$, A is infinite, and $A \leq_T B$ whenever B is an infinite subset of A. Here $\leq_T$ means Turing reducible to and $=_T$ means Turing equivalent to. Such a set A is easy to construct. We actually only need A to be Δ^1_1 in every infinite subset of itself. Let $\eta : A \to \omega$ be a function such that $(\forall n \in \omega)\, \eta^{-1}(n)$ is infinite. Consider an arbitrary $x = \langle x_0, x_1, ...\rangle \in {}^\omega\omega$. Let $i_0 < i_1 < ...$ be the sequence of indices listing which numbers x_i are in A. That is, each $x_{i_k} \in A$, but no other x_i is in A. Define

$$f_a(x) := \langle \eta(x_{i_0}), \eta(x_{i_1}), ...\rangle$$

If there are only finitely many x_i in A, define $f_a(x)$ to be anything.

One can check that the function f_a is Baire class one (the pointwise limit of the sequence of continuous functions). One might wonder if we could define f_a differently to be continuous but still encode a in the sense that given any Borel $g : {}^\omega\omega \to {}^\omega\omega$ satisfying $f_a \cap g = \emptyset$, a is in some countable set associated to g. The answer is no for the reason that the cofinality of the poset of all continuous functions from ${}^\omega\omega$ to ω ordered by everywhere domination is $\mathfrak{d}$, the dominating number, which can be consistently less than 2^ω.

4. Reachability

In this section we introduce some combinatorial lemmas needed for the main theorem. The results may be of independent interest to the reader.

Definition 4.1. Fix $h : {}^{<\omega}\omega \to \omega$, $A \subseteq \omega$, and $t_1, t_2 \in {}^{<\omega}\omega$. We write

$$t_2 \sqsupseteq_h t_1$$

and say that t_2 is an extension of t_1 *to the right* of h iff $t_2 \sqsupseteq t_1$ and $(\forall n \in \mathrm{Dom}(t_2) - \mathrm{Dom}(t_1))\, t_2(n) \geq h(t_2 \restriction n)$. We write

$$t_2 \sqsupseteq^A t_1$$

iff $t_2 \sqsupseteq t_1$ and $(\forall n \in \mathrm{Dom}(t_2) - \mathrm{Dom}(t_1))\, t_2(n) \notin A$. We write

$$t_2 \sqsupseteq^A_h t_1$$

iff both $t_2 \sqsupseteq_h t_1$ and $t_2 \sqsupseteq^A t_1$.

Definition 4.2. Given $h_1, h_2 : {}^{<\omega}\omega \to \omega$, we write $h_1 \leq h_2$ iff

$$(\forall t \in {}^{<\omega}\omega)\, h_1(t) \leq h_2(t).$$

The following notion is crucial for the ability to find $\sqsupseteq^A$ extensions of a node t in a set $S \subseteq {}^{<\omega}\omega$.

Definition 4.3. Given $t \in {}^{<\omega}\omega$ and $S \subseteq {}^{<\omega}\omega$,

- t is 0-S-*reachable* iff $t \in S$;
- for $\alpha > 0$, t is α-S-*reachable* iff t is β-S-reachable for some $\beta < \alpha$ or $\{n \in \omega : (\exists \beta < \alpha)\, t^\frown n \text{ is } \beta\text{-}S\text{-reachable}\}$ is infinite.

- t is *S-reachable* iff t is α-S-reachable for some α.

A computation shows the following:

- t is S-reachable iff t is α-S-reachable for some $\alpha < \omega_1^{CK}(S)$.
- Given $\alpha < \omega_1^{CK}$, the set of all t that are β-S-reachable for some $\beta < \alpha$ is $\Delta^1_1(S)$.

Lemma 4.4 (Reachability Dichotomy)**.** *Fix $t \in {}^{<\omega}\omega$, $S \subseteq {}^{<\omega}\omega$, and $A \subseteq \omega$ which is infinite and Δ^1_1 in every infinite subset of itself. Assume $A \notin \Delta^1_1(S)$.*

- *If t is not S-reachable, then*
$$(\exists h \in \Delta^1_1(S))(\forall t' \sqsupseteq_h t)\, t' \notin S.$$
- *If t is S-reachable, then*
$$(\forall h)(\exists t' \sqsupseteq^A_h t)\, t' \in S.$$

Proof. First, consider the case that t is not S-reachable. If $\tilde{t}$ is a node which is not S-reachable, then there must be only finitely many $\tilde{t}^\frown n$ that are S-reachable. For each $\tilde{t}$ that is not S-reachable, define $h(\tilde{t})$ to be the smallest n such that $(\forall m \ge n)\, \tilde{t}^\frown m$ is not S-reachable. For each $\tilde{t}$ that is S-reachable, define $h(\tilde{t}) = 0$. A computation shows that $h \in \Delta^1_1(S)$. This function h witnesses that $(\forall t' \sqsupseteq_h t)\, t' \notin S$.

Consider the second case that t is S-reachable. Fix t,S, and A as in the statement of the lemma. Assume that t is S-reachable and fix $h : {}^{<\omega}\omega \to \omega$. We must find some $t' \sqsupseteq^A_h t$ such that $t' \in S$.

Assume that t is not 0-S-reachable, otherwise we are already done by setting $t' = t$. Thus, fix the smallest $\alpha > 0$ such that t is α-S-reachable.

By induction, it suffices to find some $n \in \omega$ such that $n \notin A$, $n \ge h(t)$, and $t^\frown n$ is β-S-reachable for some $\beta < \alpha$. That is, if we keep doing this, then we will have a decreasing sequence of ordinals $\alpha_0 > \alpha_1 > ...$ which must eventually reach 0, at which point we will be done. Let
$$B := \{n \in \omega : (\exists \beta < \alpha)\, t^\frown n \text{ is } \beta\text{-}S\text{-reachable}\}.$$
B is infinite and $B \in \Delta^1_1(S)$. If $B - A$ is infinite, we can get the desired n. Now, $B - A$ must be infinite because otherwise $B \cap A =_T B$ and $B \cap A$ is infinite, so
$$A \le_{\Delta^1_1} B \cap A =_T B \le_{\Delta^1_1} S,$$
which implies $A \le_{\Delta^1_1} S$, a contradiction. $\square$

5. Main Theorem

We will prove the main theorem by using a variant of Hechler forcing. In fact, we could have used a slight variant of Hechler focing where the functions in the conditions are required to be strictly increasing (see [1]). However, we thought the Reachability Dichotomy (Lemma 4.4) was worth presenting for its own sake, and that lemma encapsulates the relevant rank analysis corresponding to what was carried out in [1].

Definition 5.1. $\mathbb{H}$ is the poset of all pairs (t, h) such that $t \in {}^{<\omega}\omega$ and $h : {}^{<\omega}\omega \to \omega$, where $(t_2, h_2) \le (t_1, h_1)$ iff $t_2 \sqsupseteq_{h_1} t_1$ and $h_2 \ge h_1$. Given $A \subseteq \omega$, we write $(t_2, h_2) \le^A (t_1, h_1)$ iff $t_2 \sqsupseteq^A_{h_1} t_1$ and $h_2 \ge h_1$.

From the Reachability Dichotomy follows the Main Lemma. Recall that $(\forall x, y \in {}^\omega\omega)\, x \in \Delta^1_1(y)$ iff every ω-model M which contains y also contains x.

Lemma 5.2 (Main Lemma)**.** *Let M be an ω-model of ZF and $U \in \mathcal{P}^M(\mathbb{H}^M)$ be a set dense in $\mathbb{H}^M$. Let $A \subseteq \omega$ be infinite and Δ^1_1 in every infinite subset of itself but $A \notin M$. Then*

$$(\forall p \in \mathbb{H}^M)(\exists p' \le^A p)\, p' \in U.$$

Proof. Define

$$S := \{t \in {}^{<\omega}\omega : (\exists h \in M)\, (t, h) \in U\}.$$

We have $S \in M$. It must be that $A \notin \Delta^1_1(S)$, because otherwise since M is an ω-model, we would have $A \in M$.

Now fix an arbitrary $p = (t, h) \in \mathbb{H}^M$. We must find some $p' = (t', h') \le^A (t, h)$ such that $p' \in U$ (and so $h' \in M$). It suffices to find some $t' \in S$ such that $t' \sqsupseteq^A_h t$.

There are two cases: t is S-reachable or not. If t is not S-reachable, then by the Reachability Dichotomy (Lemma 4.4) there is $h \in \Delta^1_1(S)$ such that $(\forall t' \sqsupseteq_h t)\, t' \notin S$. Since M is an ω-model and $S \in M$, such an h would be in M. Unpacking the definition of S, we get that U is not dense in $\mathbb{H}^M$, a contradiction.

The other case is that t is S-reachable. Lemma 4.4 gives us a $t' \in S$ such that $t' \sqsupseteq^A_h t$, which is what we wanted. □

This next theorem refers to the function f_a defined in Section 3.

Theorem 5.3 (Main Theorem)**.** *Let Γ be the pointclass of all sets defined by formulas in a certain class (so it makes sense to talk about Γ-formulas). Let $\prec$ be an ordering on ${}^\omega\omega$ such that whenever $c, a \in {}^\omega\omega$ are such that $a \not\prec c$, then there exists an ω-model M of ZF such that*

- *$c \in M$;*

- *$a \notin M$;*
- *$\mathcal{P}^M(\mathbb{H}^M)$ is countable (in V);*
- *for every forcing extension N (in V) of M by $\mathbb{H}^M$, the truth (in V) of Γ formulas with real parameters in N can be computed in N.*

Then for any $a \in {}^\omega\omega$ and $g : {}^\omega\omega \to {}^\omega\omega$ in Γ,

$$f_a \cap g = \emptyset \quad \Rightarrow \quad a \prec (\textit{any code for } g).$$

Proof. Fix a, g, and an arbitrary code c for g. In any model N which contains c and which can compute the truth (in V) of Γ formulas with real parameters in N, let $\tilde{g}$ refer to the function $g \cap N$ (which is in N). Suppose $a \not\prec c$. Fix an ω-model M as in the hypothesis of the theorem. Let $A \subseteq \omega$ be the set from the definition of f_a that is Δ^1_1 in every infinite subset of itself and $a =_T A$. Note that $A \notin M$.

We will construct an $x \in {}^\omega\omega$ satisfying $f_a(x) = g(x)$ and this will prove the theorem. Let

$$\langle U_n \in \mathcal{P}^M(\mathbb{H}^M) : n < \omega \rangle$$

be an enumeration (in V) of the dense subsets of $\mathbb{H}^M$ in M. Let $\dot{x}$ be the canonical name for the generic real added by $\mathbb{H}^M$. We will construct a decreasing sequence of conditions of $\mathbb{H}^M$ which hit each U_n. The $x \in {}^\omega\omega$ will be the union of the stems in this sequence (and it will be generic over M having the name $\dot{x}$).

Starting with $1 \in \mathbb{H}^M$, apply the Lemma 5.2 to get $p_0 \leq^A 1$ in U_0. Then, apply Lemma 5.2 again to get $p_0' \leq^A p_0$ and $m_0 \in \omega$ such that $(p_0' \Vdash \tilde{g}(\dot{x})(0) = \check{m}_0)^M$. Next, extend the stem of p_0' by one to get $p_0'' \leq p_0'$ to ensure that $f_a(x)(0) = m_0$.

Next, get $p_1'' \leq p_1' \leq^A p_1 \leq^A p_0''$ such that $p_1 \in U_1$, $(p_1' \Vdash \tilde{g}(\dot{x})(1) = \check{m}_1)^M$ for some $m_1 \in \omega$, and p_1'' extends the stem of p_1' by one to ensure that $f_a(x)(1) = m_1$. Continue forever like this.

The x we have constructed is generic for $\mathbb{H}^M$ over M. Let $N = M[x]$. For each $n \in \omega$ we have $(\tilde{g}(x)(n) = m_n)^N$. Since Γ-formulas are absolute between N and V, for each $n \in \omega$ we have

$$g(x)(n) = m_n.$$

On the other hand, for each $n \in \omega$ we have $f_a(x)(n) = m_n$. □

In the following, $\mathcal{M}_n(y)$ refers to the cannonical proper class model with n Woodin cardinals which contains $y \in {}^\omega\omega$. For each $n \in \omega$ and $y \in {}^\omega\omega$, ${}^\omega\omega \cap \mathcal{M}_n(y)$ is countable. When we write $a \in \mathcal{M}_n(c)$, we will be making the assumption that $\mathcal{M}_n(c)$ exists, which has large cardinal strength.

Corollary 5.4. *Fix $a \in {}^\omega\omega$, Γ, $g : {}^\omega\omega \to {}^\omega\omega$ in Γ, and a code c for g. Assume $f_a \cap g = \emptyset$.*

- $\Gamma = \boldsymbol{\Delta}^1_1 \Rightarrow a \in \Delta^1_1(c)$;
- $\Gamma = \boldsymbol{\Delta}^1_2 \Rightarrow a \in L(c)$;
- $\Gamma = \boldsymbol{\Delta}^1_3 \Rightarrow a \in \mathcal{M}_1(c)$;
- $\Gamma = \boldsymbol{\Delta}^1_4 \Rightarrow a \in \mathcal{M}_2(c)$;
- ...

Proof. The first bullet holds because Δ^1_1 formulas are absolute between ω-models and V, and whenever $a \notin \Delta^1_1(r)$, there is some ω-model of ZF which contains r but not a. The second bullet holds by Shoenfield's Absoluteness Theorem. The last two bullets hold because a forcing extension of $\mathcal{M}_{3+n}$ below its bottom Woodin cardinal can compute the truth of $\boldsymbol{\Delta}_{1+n}$ formulas with real parameters in N. For more information related to the last two bullets, see Lemma 4.6 of [Steel]. □

From the top bullet of this corollary, it follows that there is a morphism from $\mathcal{D}_{\boldsymbol{\Delta}^1_1}$ to $\langle {}^\omega\omega, {}^\omega\omega, \leq_{\Delta^1_1} \rangle$. From the second bullet, it follows that there is a morphism from $\mathcal{D}_{\boldsymbol{\Delta}^1_2}$ to $\langle {}^\omega\omega, {}^\omega\omega, \leq_L \rangle$, etc.

6. Necessity of Hypotheses

Let $\Gamma = \bigcup_{n \in \omega} \boldsymbol{\Delta}^1_n$ be the pointclass of projective sets. By Corollary 5.4, if $g : {}^\omega\omega \to {}^\omega\omega$ is a projective function and $f_a \cap g = \emptyset$, then $a \in \bigcup_{n<\omega} \mathcal{M}_n(c)$ where c is any code for g. This implies that $||\mathcal{D}_\Gamma|| = 2^\omega$. It is natural to ask whether $||\mathcal{D}_\Gamma|| = 2^\omega$ can be proved in ZFC alone (the assumption that the $\mathcal{M}_n(c)$ exist goes far beyond ZFC). We can ask the following stronger question:

Question 6.1. Does ZFC prove that for each projective $g : {}^\omega\omega \to {}^\omega\omega$ there is a countable set $G(g) \subseteq {}^\omega\omega$, and for each $a \in {}^\omega\omega$ there is a projective function $f_a : {}^\omega\omega \to {}^\omega\omega$ such that $(\forall a \in {}^\omega\omega)(\forall g)$

$$f_a \cap g = \emptyset \Rightarrow a \in G(g)?$$

We do not know how to answer the above question. The problem is that the functions f_a for various a may have nothing to do with one another. We can, however, answer the following:

Question 6.2. Does ZFC prove that there exist functions f_a and countable sets $G(g)$ as in the above question but with the additional requirement that the mapping $(a, x) \mapsto f_a(x)$ is projective?

We will now argue that the answer to Question 6.2 is no. It suffices to show that ZFC does not prove there is a pair of mappings $a \mapsto f_a$

and $g \mapsto G(g)$ such that $(a,x) \mapsto f_a(x)$ is projective and $(\forall a \in {}^\omega\omega)(\forall g)$

$$(\forall x \in {}^\omega\omega)\, f_a(x) \leq^* g(x) \Rightarrow a \in G(g),$$

because the pointwise eventual domination relation is above the disjointness relation.

Consider a model of the following statements:

1) There is a projective wellordering of the reals of ordertype 2^ω;
2) $\neg$CH;
3) $\mathfrak{b} = 2^\omega$.

Statement 3) is equivalent to saying that each subset of ${}^\omega\omega$ of size $< 2^\omega$ is $\leq^*$-dominated by a single element of ${}^\omega\omega$. The construction of a model in which MA + $\neg$CH holds (and therefore $\mathfrak{b} = 2^\omega$) and there is a projective wellordering of the reals is done in [4]. Consider a given encoding $a \mapsto f_a$ such that the map $(a,x) \mapsto f_a(x)$ is projective. The mapping which takes $a \in {}^\omega\omega$ to a code for f_a is projective. Let $\prec$ be the projective wellordering given by 1). For each $b \in {}^\omega\omega$, we may define the function $g_b : {}^\omega\omega \to {}^\omega\omega$ as follows:

$$g_b(x) := \text{ the } \prec\text{-least } y \in {}^\omega\omega \text{ such that } (\forall a \prec b)\, f_a(x) \leq^* y.$$

Note that the prewellordering $\prec$ is used twice. Because $\mathfrak{b} = 2^\omega$, this function is indeed well-defined. It is also projective. Now, consider a set $\mathcal{A} \subseteq \mathcal{P}({}^\omega\omega)$ of size ω_1. Since $\neg$CH, we may fix a single b satisfying $(\forall a \in \mathcal{A})\, a \prec b$. By definition of g_b, we have

$$(\forall a \in \mathcal{A})(\forall x \in {}^\omega\omega)\, f_a(x) \leq^* g_b(x).$$

On the other hand, given the countable set $G(g_b) \subseteq \mathcal{P}({}^\omega\omega)$, it cannot be that $\mathcal{A} \subseteq G(g_B)$. Hence, the encoding is not as required.

7. A Forcing Free Proof

In Corollary 5.4 we showed that if $g : {}^\omega\omega \to {}^\omega\omega$ is Borel and c is any code for g, then

$$f_a \cap g = \emptyset \Rightarrow a \in \Delta^1_1(c),$$

where f_a is defined in Section 3. In this section we will present a different and forcing free proof that

$$f_a \cap g = \emptyset \Rightarrow a \in \Sigma^2_1(c).$$

To avoid complications, we will actually consider functions from ${}^\omega\omega$ to ${}^\omega 2$. The function f_a can be modified into a function from ${}^\omega\omega$ to ${}^\omega 2$ by simply replacing $\eta : A \to \omega$ with $\eta : A \to 2$ in the original definition of f_a. We will prove the desired result by proving the contrapositive. That is, fix $a \in {}^\omega\omega$, Borel $g : {}^\omega\omega \to {}^\omega 2$, and a code $c \in {}^\omega\omega$ for g. Fix $A \subseteq \omega$ that is Turing equivalent to a and A is computable from every infinite subset of itself. Assume that $a \notin \Sigma^2_1(c)$. We must construct an $x \in {}^\omega\omega$ such that

$$f_a(x) = g(x).$$

The following game theoretic notion is how we will get a forcing free proof:

Definition 7.1. Given a function $j : {}^\omega\omega \to 2$, and an $m \in 2$, $\mathcal{G}(j,m)$ is the game where Player I plays a pair $(t,h) \in \mathbb{H}$ that is $\le$ the current pair and Player II plays a pair $(t,h) \in \mathbb{H}$ that is $\le^A$ the current pair. After infinitely many moves, let $x \in {}^\omega\omega$ be the union of the first elements of the pairs played. Player II wins iff $j(x) = m$. We say that (t,h) *ensures* that $j(x) = m$ iff Player II has a winning strategy for $\mathcal{G}(j,m)$ where the starting position is (t,h).

Lemma 7.2. *If for each $i \in \omega$ and $(t,h) \in \mathbb{H}$ there exists $m \in 2$ and $(t',h') \le^A (t,h)$ which ensures $g(x)(i) = m$, then there exists an $x \in {}^\omega\omega$ such that $f_a(x) = g(x)$.*

Proof. Our x will be the union of the first elements of the pairs in the sequence we will construct. Start with the condition $(\emptyset, h) \in \mathbb{H}$ where h is arbitrary. Let $m_0 \in 2$ and $(t_0,h_0) \le^A (t,h)$ be such that (t_0,h_0) ensures $g(x)(0) = m_0$. Fix a winning strategy η_0 for Player II for the corresponding game. Have Player II play according to η_0 for one move to get $(t_0', h_0') \le^A (t_0,h_0)$. Extend t_0' by one to get $(t_0'', h_0') \le (t_0', h_0')$ so that $f_a(x)(0) = m_0$.

Let $m_1 \in 2$ and $(t_1,h_1) \le^A (t_0'', h_0')$ be such that (t_1,h_1) ensures $g(x)(1) = m_1$. Fix a winning strategy η_1 for Player II for the corresponding game. Have Player II play according to η_0 for one more and according to η_1 for one more (in the correct games) to get $(t_1', h_1') \le^A (t_1,h_1)$. Extend t_1' by one to get $(t_1'', h_1') \le (t_1', h_1')$ so that $f_a(x)(1) = m_1$. Continue like this forever. □

Once the next lemma is proved, we will be done.

Lemma 7.3. *Assuming $a \notin \Sigma^2_1(c)$, for each Borel $j : {}^\omega\omega \to 2$ and $(t,h) \in \mathbb{H}$, there exists $m \in 2$ and $(t',h') \le^A (t,h)$ which ensures $j(x) = m$.*

Proof. This can be proved by induction on the rank of j within the Baire hierarchy. The base case is when j is continuous, and the proof is immediate. For the induction step, assume that $\langle j_n : n \in \omega \rangle$ is a sequence of Borel functions such that

$$(\forall x \in {}^\omega\omega)\, j(x) = \lim_{n\to\infty} j_n(x).$$

Assume that for each $n \in \omega$ and $(\tilde{t}, \tilde{h}) \in \mathbb{H}$, there exists $m' \in 2$ and $(\tilde{t}', \tilde{h}') \leq^A (\tilde{t}, \tilde{h})$ which ensures $j_n(x) = m'$.

Let $n_0 = 0$. Let $m_0 \in 2$ and $(t_0, h_0) \leq^A (t, h)$ ensure $j_{n_0}(x) = m_0$. Let η_0 be a winning strategy for Player II for $\mathcal{G}(j_{n_0}, m_0)$. The strategy η_0 should be applied infinitely often for the remainder of the construction (assuming it does not terminate).

For $n \in \omega$ and $m \in 2$, let $S(n, m) \subseteq {}^{<\omega}\omega$ be the following set:

$$S(n, m) := \{t' \in {}^{<\omega}\omega : (\exists n' \geq n)(\exists h')\,(t', h') \text{ ensures } j_{n'}(x) = m\}.$$

There are two cases: either t_0 is $S(n_0 + 1, 1 - m_0)$-reachable or not. First, assume that it is not. We may fix $\tilde{h} \geq h$ from Lemma 4.4 such that $(\forall t' \sqsupseteq_{\tilde{h}} t_0)\, t' \notin S(n_0 + 1, 1 - m_0)$. We claim that $(t_0, \tilde{h})$ ensures $j(x) = m_0$. To see why, consider the following strategy of Player II: 1) make $\leq^A$-extensions to either ensure the value of $j_n(x)$ for all $n \leq 1$ (and these values can only be ensured to be m_0), and 2) periodically play according to the winning strategies being produced from the ensuring process. When the game finishes, calling x the real constructed, $j_n(x) = m_0$ for all $n \geq n_0$, and so also $j(x) = m_0$.

The other case is that t_0 is $S(n_0+1, 1-m_0)$-reachable. It is important that t_0 can reach $S(n_0 + 1, 1 - m_0)$ by making a $\leq^A$-extension, instead of an arbitrary $\leq$-extension. The set $S(n_0+1, 1-m_0)$ is $\Sigma^2_1(c)$ (because the definition of the set existentially quantifies over winning strategies for a game of real information). It cannot be that A is Σ^2_1 in $S(n_0 + 1, 1 - m_0)$, because if it was then by transitivity we would have that a is $\Sigma^2_1(c)$. Since A is not Σ^2_1 in $S(n_0 + 1, 1 - m_0)$, it is also not Δ^1_1 in it, so by Lemma 4.4 we may fix $(t_0', h_0) \leq^A (t_0, h_0)$ such that $t_0' \in S(n_0 + 1, 1 - m_0)$. At this point, apply the strategy η_0 one time to get $(t_0'', h_0'') \leq^A (t_0', h_0)$. Since $t_0' \in S(n_0 + 1, 1 - m_0)$, get $n_1 > n_0$, $m_1 = 1 - m_0$, and $(t_1, h_1) \leq^A (t_0'', h_0'')$ that ensures $j_{n_1}(x) = m_1$. Let η_1 be a winning strategy for Player II for $\mathcal{G}(j_{n_1}, m_1)$. The strategy η_1, along with η_0, should be applied infinitely often for the remainder of the construction (assuming it does not terminate).

There are now two cases: either t_1 is $S(n_1 + 1, 1 - m_1)$-reachable or not. If not, then we are done by reasoning similar to before. If t_1 is $S(n_1 + 1, 1 - m_1)$-reachable, then we continue the construction and the question becomes whether it ever terminates. Suppose, towards a contradiction, that the construction does not terminate. Let $x \in {}^{\omega}\omega$ be the sequence that has been constructed. For all $i \in \omega$ we have $j_{n_i}(x) = m_i$. However, the m_i's alternate, so the limit $\lim_{n\to\infty} j_n(x)$ cannot exist, which is a contradiction. $\square$

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