Aspects of Generalized Dominating Numbers

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Basic Definitions

Given a poset $\langle P, \leq_P \rangle$, define

$\text{cf}\langle P, \leq_P \rangle := \min\{|A| : (\forall p \in P)(\exists a \in A)\, p \leq a\}$.

Given a set Γ of functions from a set X to a regular cardinal κ,

$$\mathfrak{d}(\Gamma) := \text{cf}\langle \Gamma, \leq \rangle,$$

where $f \leq g$ iff $(\forall x \in X)\, f(x) \leq g(x)$.

A challenge-response relation is a tripple

$$\mathcal{R} = \langle R_-, R_+, R \rangle$$

such that $R \subseteq R_- \times R_+$. The norm $||\mathcal{R}||$ of $\mathcal{R}$ is

$$||\mathcal{R}|| := \min\{|A| : (\forall r \in R_-)(\exists a \in A)\, rRa\}.$$

Note: $\mathfrak{d}(\Gamma) = ||\langle \Gamma, \Gamma, \leq \rangle||$.

Examples

- $\mathfrak{d}({}^{\omega}\omega) = \mathfrak{d}$, the "dominating number".
- $\mathfrak{d}({}^{\lambda}\lambda)$ is similar to $\mathfrak{d}$.
- $\mathfrak{d}({}^{\omega_1}\omega) = ?$
- $\mathfrak{d}({}^{\lambda}\omega) = 2^{\lambda} = \mathfrak{c}$ when $\lambda < \mathfrak{c}$ and $\mathfrak{c}$ is real-valued measurable.
- $\mathfrak{d}({}^{\lambda}\kappa) < 2^{\lambda} = \mathfrak{c}$ when $\omega < \kappa \leq \lambda < \mathfrak{c}$ and $\mathfrak{c}$ is real-valued measurable.
- $\mathfrak{d}({}^{\mathfrak{c}}\omega) = 2^{\mathfrak{c}}$. More generally:
- $\mathfrak{d}({}^{\lambda}\kappa) = 2^{\lambda}$ when $\lambda^{\kappa} = \lambda$.

Baire Hierarchy

- Let $\mathcal{B}_0$ be the set of continuous functions from ${}^{\omega}\omega$ to ω.
- Let $\mathcal{B}_1$ be the set of functions that are limits of countable sequences of $\mathcal{B}_0$ functions. These are the "Baire class 1" functions.
- Let $\mathcal{B}_2$ be the set of functions that are limits of countable sequences of $\mathcal{B}_1$ functions, etc.

What is $\mathfrak{d}(\mathcal{B}_0)$?
What is $\mathfrak{d}(\mathcal{B}_1)$?
What is $||\langle \mathcal{B}_1, \mathcal{B}_2, \leq \rangle||$? Etc.

Well-founded Trees

Given $\alpha \leq \omega_1$, let $\mathcal{W}_\alpha$ be the set of all well-founded subtrees of ${}^{<\omega}\omega$ of rank $< \alpha$. For every $f \in \mathcal{B}_0$ there is a well-founded tree $\phi^-(f) \subseteq {}^{<\omega}\omega$, and for every well-founded tree $T \subseteq {}^{<\omega}\omega$ there is a $\phi^+(T) \in \mathcal{B}_0$ such that $(\forall f)(\forall T)$

$$\phi^-(f) \subseteq T \Rightarrow f \leq \phi^+(T).$$

Theorem 1

For every α such that $\omega \leq \alpha < \omega_1$,

$$\text{cf}\langle \mathcal{W}_\alpha, \subseteq \rangle = \mathfrak{d}.$$

Thus, $\mathfrak{d}(\mathcal{B}_0) = \mathfrak{d}$.

Constructibility

Functions from a big cardinal to a small one can encode information. Sometimes, "constructibility can be reduced to domination":

Theorem 2a

Let λ and κ be infinite cardinals. For every $A \subseteq \lambda$ there is a function $f : {}^{\kappa}\lambda \to \kappa$ such that whenever M is a transitive model of ZF such that ${}^{\kappa}\lambda \subseteq M$ and some $g : {}^{\kappa}\lambda \to \kappa$ in M dominates f, then $A \in M$.

As a consequence, this gives a new proof that $\mathfrak{d}({}^{\lambda}\kappa) = 2^{\lambda}$ when $\lambda^{\kappa} = \lambda$.
A slight variation (but only works for $\kappa = \omega$):

Theorem 2b

Let λ be an infinite cardinal. For every $A \subseteq \lambda$ there is a function $f : {}^{\omega}\lambda \to \omega$ such that whenever M is a transitive model of ZF such that $\lambda \in M$ and some $g : ({}^{\omega}\lambda)^M \to \omega$ in M satisfies

$$(\forall x \in ({}^{\omega}\lambda)^M)\, f(x) \leq g(x),$$

then $A \in M$.

Baire Class One Functions

The situation with $\mathcal{B}_1$ is very different from $\mathcal{B}_0$. We have $\mathfrak{d}(\mathcal{B}_1) = \mathfrak{c}$. In fact, if $\mathcal{F}$ is the set of *all* functions from ${}^{\omega}\omega$ to ω, then

$$||\langle \mathcal{B}_1, \mathcal{F}, \leq \rangle|| = \mathfrak{c}.$$

This follows because when $\lambda = \omega$ in Theorem 2b, the function f is in $\mathcal{B}_1$.

Weak Compactness

Theorem 2b works by using the fact that well-foundedness is absolute. If $\kappa > \omega$, we can use weak compactness to get enough absoluteness to prove an analogous result:

Theorem 2c

Let κ be an infinite cardinal. For every $A \subseteq \kappa$ there is a function $f : {}^{\kappa}2 \to \kappa$ such that whenever M is a transitive model of ZF such that $\kappa \in M$, ${}^{<\kappa}2 \subseteq M$, ($\kappa$ is weakly compact)^M, and some $g : ({}^{\kappa}2)^M \to \kappa$ in M satisfies

$$(\forall x \in ({}^{\kappa}2)^M)\, f(x) \leq g(x),$$

then $A \in M$.

Classical Results

An old result of Jockusch and Solovay is the following:

- For every Δ^1_1 set $A \subseteq \omega$ there is a function $f : \omega \to \omega$ such that if $g : \omega \to \omega$ everywhere dominates f, then $A \leq_T g$.

This contrasts with Theorem 2(a,b,c):

- For every $A \subseteq \omega$ there is a function $f : \mathbb{R} \to \omega$ such that if $g : \mathbb{R} \to \omega$ everywhere dominates f, then $A \in L[g]$.
- For every $A \subseteq \mathbb{R}$ there is a function $f : \mathbb{R} \to \omega$ such that if $g : \mathbb{R} \to \omega$ everywhere dominates f, then $A \in L(\mathbb{R}, g)$.
- For every $A \subseteq \lambda$ there is a function $f : {}^{\kappa}\lambda \to \kappa$ such that if $g : {}^{\kappa}\lambda \to \kappa$ everywhere dominates f, then $A \in L({}^{\kappa}\lambda, g)$.

Elementary Substructures

A different trick than the one in Theorem 2b and the one in Theorem 2c is to use an elementary substructure:

Theorem 2d

Let λ and κ be infinite cardinals. For every $A \subseteq \lambda$ there is a function $f : {}^{\kappa}\lambda \to \kappa$ such that whenever $\langle M, \in \rangle \prec V$ is such that ${}^{<\kappa}\lambda \subseteq M$ and some $g : {}^{\kappa}\lambda \to \kappa$ in M satisfies

$$(\forall x \in {}^{\kappa}\lambda)\, f(x) \leq g(x),$$

then $A \in M$.

Theorems 2 a,b,c,d are all incomparable. Is there a unifying result?

Eventual Domination

A different but related problem is to consider *pointwise eventual domination*. Using a significantly different argument we get the following:

Theorem 3

For every $A \subseteq \omega$ there is a function $f : {}^{\omega}\omega \times \omega \to \omega$ in $\mathcal{B}_1$ such that whenever $g : {}^{\omega}\omega \times \omega \to \omega$ in $\mathcal{B}_1$ satisfies

$$(\forall r \in {}^{\omega}\omega)(\exists c \in \omega)\, f(r, c) \leq g(r, c),$$

then A is Π^1_1 in a code for g.

This result can probably be improved by only requiring g to be Borel (or even less assuming some axioms).

Contact Information

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