Improving a result of Mostowski: Amalgamation Failure

Core Model Seminar

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Overview

Theorem (Mostowski 1976)

Let M be a transitive model of ZFC *such that* $(2^\omega)^M$ *is countable. Let* $x \in \mathbb{R}$*. Then there exist* $y, z \in \mathbb{R}$ *that are both generic over* M *such that* x *is computable from* y *and* z *together.*

Here is the focus of this presentation:

Theorem (Generic Coding with Help 2019)

Let M be a transitive model of ZFC *such that* $(2^{2^\omega})^M$ *is countable. Let* $x \in \mathbb{R}$*. Let* $y \in \mathbb{R} - M$*. Then there exists some* $z \in \mathbb{R}$ *that is generic over* M *such that* x *is computable from* y *and* z *together.*

The second theorem comes from [2] and [3] and many applications of the theorem are explored in [4].

Proof of Mostowski's Theorem

Let M be a transitive model of ZFC such that $(2^\omega)^M$ is countable. Fix $x \in {}^\omega 2$. We will find $y, z \in {}^\omega 2$ Cohen generic over M such that $x = y$ XOR z.

Let $\langle D_n : n < \omega \rangle$ be an enumeration of all dense subsets (in M) of Cohen forcing.

Let $y_0 \in {}^{<\omega}2$ be in D_0. Let $z_0 \in {}^{<\omega}2$ be the same length as y_0 such that

$$y_0 \text{ XOR } z_0 = x \restriction |y_0|.$$

Let $z_1 \sqsupseteq z_0$ be in D_0. Let $y_1 \sqsupseteq y_0$ be the same length as z_1 such that

$$y_1 \text{ XOR } z_1 = x \restriction |y_1|.$$

Let $y_2 \sqsupseteq y_1$ be in D_1. Let $z_2 \sqsupseteq z_1$ be the same length as y_2 such that

$$y_2 \text{ XOR } z_2 = x \restriction |y_2|.$$

Continue like this. The reals $y = \bigcup y_i$ and $z = \bigcup z_i$ are as desired.

More complicated patterns with Cohen reals

Theorem (Mostowski)

Let M be a c.t.m. of ZFC. Fix $l \in \omega$. Let $\mathcal{A}$ be a collection of subsets of $\{0, \ldots, l\}$ that contains all singletons and is closed under subsets. Fix any $x \in {}^{\omega}2$. Then there are reals $g_0, \ldots, g_l$ all Cohen generic over M such that for any $A \subseteq \{0, \ldots, l\}$,

1) if $A \in \mathcal{A}$, then $\{g_i : i \in A\}$ is contained in a forcing extension of M;
2) if $A \notin \mathcal{A}$, then $x \in L[\{g_i : i \in A\}]$.

In their paper Set Theoretic Blockchains [1], the authors prove more complicated versions of this.

Towards proving the Generic Coding with Help Theorem

Let M be a c.t.m. of ZFC. Say that $y \in {}^{\omega}2$ is **helpful** iff for any $x \in {}^{\omega}2$, there is a z generic over M such that $x \in L[y, z]$.

Theorem (Habič et al [1])

If y is Cohen generic over M, it is helpful.

Theorem (Habič, Sy Friedman)

If y is unbounded over M, it is helpful.

Theorem (Sy Friedman)

If y is Sacks generic over M, it is helpful.

The Generic Coding with Help Theorem:

Theorem [4]

If y is *any* real not in M, it is helpful.

The Secret

The attemps of both Habič and Sy Friedman to prove the Generic Coding with Help Theorem all involved Cohen forcing. It is still open whether this is possible.

The key idea is to abandon Cohen forcing and instead use *Tree Hechler Forcing*.

Main Theorem: Part 1

Recall *Tree Hechler Forcing*, whose conditions are trees with cofinite splitting beyond their stems:

Definition

The forcing $\mathbb{H}$ consists of the trees $T \subseteq {}^{<\omega}\omega$ such that for all $t \sqsupseteq \mathrm{Stem}(T)$ in T,

$$\{n \in \omega : t^\frown n \notin T\} \text{ is finite.}$$

The ordering is by inclusion.

Given a generic Z for $\mathbb{H}$, we have that Z can be recovered from $\bigcup \bigcap Z$ (the union of the stems of conditions in Z), which is a function from ω to ω.

Main Theorem: Part 2

Given a set $Y \subseteq \omega$, we can define an auxillary ordering $\leq_Y$ on $\mathbb{H}$:

Definition

Let $Y \subseteq \omega$. Given $t, t' \in {}^{<\omega}\omega$, we write $t' \sqsupseteq_Y t$ iff $t' \sqsupseteq t$ and

$$(\forall n \in \mathrm{Dom}(t') - \mathrm{Dom}(t))\, t'(n) \notin Y.$$

So if $t' \sqsupseteq_Y t$, then t' does not "hit" Y any more than t already does.

Definition

Let $Y \subseteq \omega$. Given $T, T' \in \mathbb{H}$, we write $T' \leq_Y T$ iff $T' \leq T$ and $\mathrm{Stem}(T') \sqsupseteq_Y \mathrm{Stem}(T)$.

So $T' \leq_A T$ means that T' is stronger than T and the stem of T' does not hit Y any more than the stem of T.

Main Theorem: Part 3

Given an $\mathbb{H}$-generic Z (over a transitive model) and a $Y \subseteq \omega$, here is how we can decode a real $x \in {}^{\omega}2$: let $y_0, y_1, \ldots \in \omega$ be the increasing enumeration of Y. Let $\eta_Y : Y \to 2$ be the function $\eta_Y(y_i) = 0$ if i is even, and 1 if i is odd.

Consider $\tilde{z} = \bigcup \bigcap Z$ ($\tilde{z}$ is the union of the stems of conditions in Z). We have $\tilde{z} : \omega \to \omega$. Let $\ell_0 < \ell_1 < \ldots$ be the increasing enumeration of the set of $\ell \in \omega$ such that $\tilde{z}(\ell) \in Y$. That is, the ℓ_i's are the levels where $\tilde{z}$ hits Y. We can now "decode" the real $\langle \eta_Y(\tilde{z}(\ell_i)) : i < \omega \rangle \in {}^{\omega}2$. That is, every time z hits Y, a new bit is encoded according to which element of Y is hit.

Question

Given $Y \subseteq \omega$ and a transitive model M of ZF, can we create a generic Z for $\mathbb{H}$ over M without encoding unwanted bits in the process?

By genericity, we **cannot** if $Y \in M$. But we **can** if Y is infinite but has no infinite subset in M (the Main Lemma)!

Main Theorem: Part 4

Main Lemma

Let M be a transitive model of ZF. Let $Y \subseteq \omega$ be infinite but have no infinite subset in M.

Let $\mathbb{P} = \mathbb{H}^M$. Let $\mathcal{D} \in \mathcal{P}^M(\mathbb{P})$ be open dense (in M). Let $T \in \mathbb{P}$.

Then there exists some $T' \leq_Y T$ in $\mathcal{D}$.

So if $\mathcal{P}^M(\mathbb{P})$ is countable (in V), then constructing a Z which is $\mathbb{P}$-generic over M can be accomplished by constructing a decreasing $\leq_Y$-sequence. to hit all ω many dense sets.

Assuming the Main Lemma is true, we can prove the Main Theorem.

Main Theorem: Part 5

So now we can alternate between hitting dense sets by making $\leq_Y$-extensions, and encoding whatever bits we want by making non-$\leq_Y$-extensions.

Generic Coding with Help Theorem (Main Theorem)

Let M be a transitive model of ZF such that $\mathcal{P}^M(\mathbb{H}^M)$ is countable. Let $x, y \in {}^\omega 2$ be such that $y \notin M$.

Then there is a Z that is $\mathbb{H}^M$-generic over M such that $x \in L[y, Z]$.

Proof: let $Y \subseteq \omega$ be Turing equivalent to y and also computable from every infinite subset of itself. Let $\langle \mathcal{D}_i : i < \omega \rangle$ be an enumeration of the open dense subsets of $\mathbb{H}^M$ in M. Let $T'_{-1} := 1 \in \mathbb{H}^M$. Now let $i \geq 0$.

Let $T_i \leq_Y T'_{i-1}$ be such that $T_i \in \mathcal{D}_i$. Let $T'_i \leq T_i$ be a non-$\leq_Y$-extension of T_i extending the stem of T_i by one to encode the i-th bit of x, etc.

Proving the Main Lemma: Part 1

But how do we prove the Main Lemma? That is, how do we $\leq_Y$-extend a condition to hit a dense subset $\mathcal{D}$ of $\mathbb{H}^M$ in our countable transitive model M?

Taking one step:

Sticking Out Observation

Let M be a transitive model of ZF. Let $Y \subseteq \omega$ be infinite but there are no infinite subsets of Y in M. Then if $B \subseteq \omega$ is infinite and in M, then $B - Y$ is infinite.

Proof: Assume towards a contradiction that $B - Y$ is finite. Then $B - Y \in M$. Since both B and $B - Y$ are in M, we have $B \cap Y \in M$ as well. At the same time, since B is infinite and $B - Y$ is finite, $B \cap Y$ must be infinite. So now $B \cap Y$ is an infinite subset of Y which is in M, which is a contradiction.

Proving the Main Lemma: Part 2

To prove the Main Lemma, we need a "rank analysis" on the stems of the conditions in our dense set.

Definition

Given $S \subseteq {}^{<\omega}\omega$ and $t \in {}^{<\omega}\omega$,

- t is 0-S-reachable iff $t \in S$;
- t is α-S-reachable for some $\alpha > 0$ iff

$$\{n \in \omega : t^\frown n \text{ is } \beta\text{-}S\text{-reachable for some } \beta < \alpha\}$$

  is infinite.
- t is S-reachable iff t is α-S-reachable for some α.

Fact: fix $T \in \mathbb{H}$ and $\mathcal{D} \subseteq \mathbb{H}$. Let $t = \text{Stem}(T)$ and $S \subseteq {}^{<\omega}\omega$ be the set of stems of elements of $\mathcal{D}$. Then $\mathcal{D}$ is dense below T iff t is S-reachable.

Proving the Main Lemma: Part 3

Proof of Main Lemma: fix M and $\mathcal{D} \subseteq \mathbb{H}^M$ which is dense. Fix $T \in \mathbb{H}^M$. Let $Y \subseteq \omega$ be infinite with no infinite subsets in M. We will find $T' \leq_Y T$ in $\mathcal{D}$. Let $t = \text{Stem}(T)$ and S be the set of stems of elements of $\mathcal{D}$. It suffices to find some $s \sqsupseteq_Y t$ in $T \cap S$.

Let α be such that t is α-S-reachable. If $\alpha = 0$, then set $s := t$ and we are done. If not, the set $B = \{n \in \omega : t^\frown n \text{ is } \beta\text{-}S\text{-reachable for some } \beta < \alpha\}$ is infinite (and in M). So by the Sticking Out Observation, $B - Y$ is infinite. So, fix some $n_0 \in (B - A)$ such that $t^\frown n_0 \in T$.

Now $t^\frown n_0$ is β-S-reachable for some $\beta < \alpha$. If $\beta = 0$, then set $s := t^\frown n_0$ are we are done. If not, then we can find some $n_1 \notin Y$ such that $t^\frown n_0^\frown n_1 \in T$ and $t^\frown n_0^\frown n_1$ is δ-S-reachable for some $\delta < \beta$, etc.

This process must eventually terminate.

Plan for the rest of the talk

We will give several applications of the Generic Coding with Help Theorem (and also the Main Lemma itself).

Then we will talk about how to enhance the Generic Coding with Help Theorem.

Application #1: The complement of a Turing Cone

In the Generic Coding with Help Theorem, we actually have $x \leq_T y \oplus z$ where $z = \bigcup \bigcap Z$.

Corollary

Assume $0^\#$ exists. Let S be the set of reals that are generic over L. Then although S is disjoint from the Turing cone above $0^\#$, we have that for any $y \in {}^\omega 2 - L$, the set

$$\{y \oplus z : z \in S\}$$

contains a Turing cone.

Application #2: The Mantle part 1

Given a countable transitive model M of ZFC, the **generic multiverse** of M is the smallest collection of countable transitive models of ZFC that can be obtained from M by repeatedly taking forcing extensions and grounds.

Given M, the mantle $\mathbb{M}$ of M is the smallest element of the generic multiverse of M with respect to inclusion (if it exists).

We say that two countable transitive models M_1, M_2 of ZFC of the same ordinal height **amalgamate** iff there is a countable transitive model M_3 of ZFC of the same ordinal height such that $M_1, M_2 \subseteq M_3$.

Corollary

Let M be a countable transitive model of ZFC. The mantle (if it exists) is the only universe in the generic multiverse of M that amalgamates with every universe in M's generic multiverse.

Application #2: The Mantle part 2

That corollary regarding the mantle follows from the following more general result. Given an ordinal α, the set $\mathcal{H}_\alpha$ refers to the collection of all countable transitive models of ZFC of ordinal height α.

Theorem

Let $M_1, J \in \mathcal{H}_\alpha$ be such that $M_1 \not\subseteq J$. Then there is a forcing extension M_2 of J that does not amalgamate with M_1.

Proof: let $y' \in M_1 - J$ be a set of ordinals. Force over J to get $J[G_1]$ to make $\sup y'$ countable. Now y' is encoded by a real $\bar{y}$ (which by mutual genericity can be assumed to not be in $J[G_1]$). Then force over $J[G_1]$ to get $M_2 := J[G_1][G_2]$, using the Generic Coding with Help Theorem, so that G_2 together with $\bar{y}$ computes a real not in any model of ZFC of height α.

History: I asked Friedman if the Generic Coding with Help Theorem was already known, and he said no but he was looking for such a theorem to make the proof above work.

Application #2: The Mantle part 3

Here is an immediate consequence of the previous theorem:

Corollary

Let $\alpha < \omega_1$ and suppose that L_α satisfies ZFC. Then L_α is the only countable transitive model of ZFC of height α which amalgamates with all other countable transitive models of ZFC of height α.

The fancy way of saying this is "the hyperuniverse has no non-trivial nodes of compatibility".

Of course, this should NOT be taken as a philosophical argument that V satisfies $V = L$.

Application #3: HOD

Every real is generic over HOD.

Every real is generic over HOD with help using $\mathbb{H}^{\mathrm{HOD}}$.

$\mathbb{H}$ has size 2^ω (and is c.c.c).

Question

Is every real generic over HOD by a poset of size $\leq (2^\omega)^{HOD}$?

No. This fails in certain universes (see [4]).

Application #4: Uniformizations

In addition to the Generic Coding with Help Theorem having applications, there is an application of the underlying technology (the Main Lemma). This was the original motivation for this work:

Theorem (ZF) [3]

Assume there is no injection of ω_1 into $\mathbb{R}$. Fix $y \in \mathbb{R}$. There is a Baire class one function $f_y : {}^\omega\omega \to {}^\omega\omega$ with the following property:

whenever $g : {}^\omega\omega \to {}^\omega\omega$ is ∞-Borel and $f_y \cap g = \emptyset$, then

$$y \in L[C]$$

where C is any ∞-Borel code for g.

Larger Sets are Generically Helpful

The Generic Coding with Help Theorem uses a real number as "help". Larger objects can be used as help when we force over the universe to make them countable.

Corollary (Larger Generic Coding with Help)

Let M be a transitive model of ZF. Let λ be a cardinal such that $\lambda \in M$. Let $\mathbb{P} = (\mathrm{Col}(\omega, \lambda) * \mathbb{H})^M$. Let $\tilde{V}$ be an outer model of V in which $\mathcal{P}^M(\mathbb{P})$ is countable.

Let $X \in \mathcal{P}^{\tilde{V}}(\lambda)$. Let $Y \in \mathcal{P}^{\tilde{V}}(\lambda) - M$. Then there is a Z in $\tilde{V}$ such that

1) Z is $\mathbb{P}$-generic over M,
2) $X \in L(Y, Z)$.

Application #5: Coding the Universe

The following is what happens when we combine Generic Coding with Help with Coding the Universe:

Corollary

Let M be any inner model satisfying ZFC. Let $\bar{y} \subseteq \mathrm{Ord}$ be a set of ordinals not in M. Then there is a set Z that is set generic over M (which exists in a class forcing extension of V) such that $V \subseteq L(\bar{y}, Z)$.

Proof: Let λ be the ordinal $\sup(\bar{y})$. Let $\mathbb{P} = (\mathrm{Col}(\omega, \lambda) * \mathbb{H})^M$. Let $\tilde{V}$ be a set forcing extension of V in which $\mathcal{P}^M(\mathbb{P})$ is countable. Let c be a real class generic over $\tilde{V}$ such that $\tilde{V} \subseteq L[c]$. Apply Larger Generic Coding with Help inside $\tilde{V}[c]$ to get set Z that is $\mathbb{P}$-generic over M such that $c \in L(\bar{y}, Z)$. So we have

$$V \subseteq \tilde{V} \subseteq L[c] \subseteq L(\bar{y}, Z).$$

Are Larger Sets Helpful in V itself Part 1?

The previous slide shows that larger sets are helpful in sufficiently large forcing extensions of the universe. What about in V itself?

Here is an immediate observation:

Theorem

Assume CH and a proper class of Woodin cardinals. Let $X \subseteq \mathbb{R}$. Then there are $Y, Z \subseteq \mathbb{R}$ that are both generic for countably closed forcings over $L(\mathbb{R})$ such that

$$X \in L(Y, Z, \mathbb{R}).$$

Proof: Since we are assuming CH, instead of talking about sets of reals we might as well be talking about subsets of ω_1. Then we can repeat Mostowski's theorem.

Are Larger Sets Helpful in V itself Part 2?

Conjecture

Assume CH and a proper class of Woodin cardinals. There is a countably closed poset $\mathbb{P} \in L(\mathbb{R})$ such that for any $X \subseteq \mathbb{R}$, there is some Z that is $\mathbb{P}$-generic over $L(\mathbb{R})$ such that

$$X \in L(\mathbb{R}^{\#}, Z, \mathbb{R}).$$

If this is true, probably $\mathbb{R}^{\#}$ can be replaced with *any* $Y \subseteq \mathbb{R}$ that is not in $L(\mathbb{R})$.

Note: Woodin has conjectured that if CH holds and there is a proper class of Woodins and there is a mouse with a measurable Woodin cardinal, then for any $X \subseteq \omega_1$, there is some model M of AD containing all the reals such that X is $\mathrm{Col}(\omega_1, \mathbb{R})$-generic over M.

Thank You!

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