

The Halpern-Läuchli Theorem at a Measurable Cardinal

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Basic Definitions

Definition

Let κ be a regular cardinal. A tree $T \subseteq {}^{<\kappa}\kappa$ is **regular** iff it is

- 1) perfect,
- 2) suitable (every maximal branch has length κ), and
- 3) a κ -tree (every level $T(\alpha) := T \cap {}^\alpha\kappa$ of T has size $< \kappa$).

Note: If κ is not strongly inaccessible, there are no regular trees.

Definition

Given sets $T_0, \dots, T_{d-1} \subseteq {}^{<\kappa}\kappa$, $T_0 \otimes \dots \otimes T_{d-1}$ is the set of d -tuples $\langle t_0, \dots, t_{d-1} \rangle$ such that each $t_i \in T_i$ and the t_i 's are all on the same level.

Definition

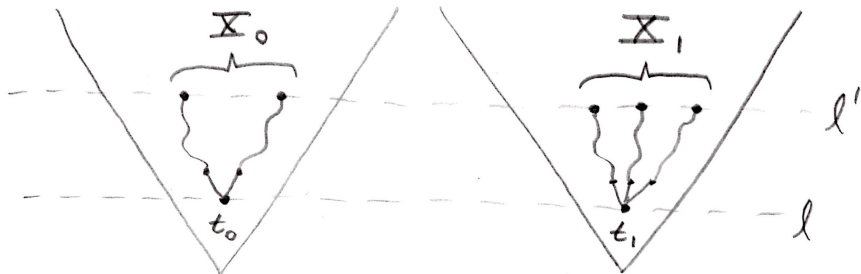
Given sets $A, X \subseteq {}^{<\kappa}\kappa$, we say that X **dominates** A iff each $a \in A$ is extended by some $x \in X$.

Definitions: SDHL

SDHL stands for “Somewhere Dense Halpern-Läuchli”.

Definition

Given a cardinal $\sigma > 0$, $\text{SDHL}(d, \sigma, \kappa)$ is the statement that given any regular trees $T_0, \dots, T_{d-1} \subseteq {}^{<\kappa}\kappa$ and any level coloring $c : T_0 \otimes \dots \otimes T_{d-1} \rightarrow \sigma$, there are levels $l < l' < \kappa$, a sequence of nodes $\langle t_i \in T_i(l) : i < d \rangle$, and a sequence of sets $\langle X_i \subseteq T_i(l') : i < d \rangle$ such that each X_i dominates $\text{Succ}_{T_i}(t_i)$ and c is constant on $X_0 \otimes \dots \otimes X_{d-1}$.



Definitions: HL

HL stands for “Strong Subtree Halpern-Läuchli”, and $\mathbf{HL}(d, \sigma, \kappa)$ is equivalent to $\mathbf{SDHL}(d, \sigma, \kappa)$ when κ is strongly inaccessible.

Definition

Given regular trees $S \subseteq T \subseteq {}^{<\kappa}\kappa$, we say that S is a **strong** subtree of T as witnessed by $A \in [\kappa]^\kappa$ iff for each $l \in \kappa$ and $t \in S(l)$,

- 1) If $l \notin A$, then $|\text{Succ}_S(t)| = 1$;
- 2) If $l \in A$, then $\text{Succ}_S(t) = \text{Succ}_T(t)$.

Definition

Given a cardinal $\sigma > 0$, $\mathbf{HL}(d, \sigma, \kappa)$ is the statement that given any regular trees $T_0, \dots, T_{d-1} \subseteq {}^{<\kappa}\kappa$ and any level coloring $c : T_0 \otimes \dots \otimes T_{d-1} \rightarrow \sigma$, there are strong trees $S_0 \subseteq T_0, \dots, S_{d-1} \subseteq T_{d-1}$ all witnessed by the same set of levels $A \in [\kappa]^\kappa$ and $(\forall l \in A)$ c is constant on $S_0(l) \otimes \dots \otimes S_{d-1}(l)$.

For the rest of this presentation, assume $0 < d < \omega$ and $0 < \sigma < \kappa$.

Complexity and reflection at a measurable

$\text{SDHL}(d, \sigma, \kappa)$ is a $\mathbf{\Pi}_1$ statement about $V_{\kappa+1}$. Let M be a model of ZF such that $V_\kappa \subseteq M$. If $\text{SDHL}(d, \sigma, \kappa)$ is true in V , then it is true in M . If $V_{\kappa+1} \subseteq M$, then the other direction holds.

$\text{HL}(d, \sigma, \kappa)$ is a $\mathbf{\Pi}_2$ statement about $V_{\kappa+1}$.

Proposition (D., H.)

Let κ be a measurable cardinal with a normal measure \mathcal{U} . Fix d and $\sigma < \kappa$. Then $\text{SDHL}(d, \sigma, \kappa)$ iff

$$\{\alpha < \kappa : \text{SDHL}(d, \sigma, \alpha)\} \in \mathcal{U}.$$

The same is true for $\text{HL}(d, \sigma, \kappa)$ in place of SDHL .

Proof: Let $j : V \rightarrow M$ be the ultrapower embedding. Because $V_{\kappa+1} \subseteq M$, $\text{SDHL}(d, \sigma, \kappa) \Leftrightarrow \text{SDHL}(d, \sigma, \kappa)^M$. By Łos's Theorem, $\text{SDHL}(d, \sigma, \kappa)^M \Leftrightarrow \{\alpha < \kappa : \text{SDHL}(d, \sigma, \alpha)\} \in \mathcal{U}$. The same argument works for $\text{HL}(d, \sigma, \kappa)$ in place of SDHL .

Upwards stationary reflection for SDHL

Proposition (D., H.)

Assume that

$$S := \{\alpha < \kappa : \text{SDHL}(d, \sigma, \alpha)\}$$

is stationary. Then $\text{SDHL}(d, \sigma, \kappa)$ holds.

Let $\langle T_i \subseteq {}^{<\kappa}\kappa : i < d \rangle$ be a sequence of regular trees and let $c : \bigotimes_{i < d} T_i \rightarrow \sigma$ be a coloring. If we can find an $\alpha < \kappa$ such that each $T_i \cap {}^{<\alpha}\kappa$ is an α -tree and $\text{SDHL}(d, \sigma, \alpha)$ holds, then we will be done. An elementary argument shows that for each $i < d$, there is a club $C_i \subseteq \kappa$ such that $(\forall \alpha \in C_i) T_i \cap {}^{<\alpha}\kappa$ is an α -tree. The set $\bigcap_{i < d} C_i$ is a club, so it must intersect S . An $\alpha < \kappa$ in the intersection is as desired.

Corollary

If $\text{SDHL}(d, \sigma, \alpha)$ holds for a stationary set of $\alpha < \kappa$, then $\text{SDHL}(d, \sigma, \kappa)$ holds in V and in any $<\kappa$ -closed forcing extension.

Proving HL

$\text{HL}(d, \sigma, \omega)$ can be proved by induction on $d < \omega$ (see [7]). The successor step involves a fusion argument. This cannot be generalized to the $\kappa > \omega$ case because the intersection of a decreasing sequence of regular trees may not be regular.

There is another proof of $\text{HL}(d, \sigma, \omega)$ (see [3]) which adds many Cohen reals by forcing, and uses an ultrafilter in the extension to make selections. This generalizes to the $\kappa > \omega$ case if we assume that κ is measurable in the extension:

Theorem (see [1])

Let $\lambda > \kappa$ satisfy $\lambda \rightarrow (\kappa)_{\kappa}^d$. Assume κ is measurable in the forcing extension where we add λ many Cohen subsets of κ . Then $\text{HL}(d, \sigma, \kappa)$ holds (in the ground model).

In [6], there is a theorem with a similar hypothesis and the conclusion implies $\text{HL}(1, \sigma, \kappa)$ for all $\sigma < \kappa$, not just finite $\sigma < \omega$ which we mentioned was true before.

Getting HL at a measurable

The cardinal κ is α -strong iff there is an elementary embedding $j : V \rightarrow M$ such that $\text{crit}(j) = \kappa$ and $V_{\kappa+\alpha} \subseteq M$. By a (slight modification of a) theorem of Woodin (see [4] for a proof), if GCH holds and κ is $(\kappa + d)$ -strong, then there is a forcing extension in which κ is measurable and remains measurable after adding $\lambda = \kappa^{+d}$ Cohen reals. This gives us the following:

Corollary

Assume there is a model in which GCH holds and there is a cardinal κ which is $(\kappa + d)$ -strong. Then there is a forcing extension in which κ is measurable and $\text{HL}(d, \sigma, \kappa)$ holds.

Question: is the existence of a $(\kappa + d)$ -strong cardinal equiconsistent with there existing a measurable κ such that $(\forall \sigma < \kappa) \text{HL}(d, \sigma, \kappa)$?

Preservation by small forcings

The following works for **HL** in place of SDHL.

Theorem (D., H.)

Let \mathbb{P} be a forcing of size $< \kappa$. Then $\text{SDHL}(d, \sigma \cdot |\mathbb{P}|, \kappa)$ implies $1 \Vdash_{\mathbb{P}} \text{SDHL}(d, \sigma, \kappa)$.

Sketch ($d = 2$ case): Given a name \dot{T} for a regular tree, let $\text{Der}(\dot{T})$ be the set of all equivalence classes of pairs $(\dot{\tau}, \alpha)$ such that

$$1 \Vdash_{\mathbb{P}} (\dot{\tau} \in \dot{T} \text{ and } \text{Length}(\dot{\tau}) = \check{\alpha}),$$

where $(\dot{\tau}_1, \alpha_1) \cong (\dot{\tau}_2, \alpha_2)$ iff $1 \Vdash_{\mathbb{P}} (\dot{\tau}_1 = \dot{\tau}_2)$. Order $\text{Der}(\dot{T})$ by $[(\dot{\tau}_1, \alpha_1)] \leq [(\dot{\tau}_2, \alpha_2)]$ iff $1 \Vdash_{\mathbb{P}} \dot{\tau}_1 \sqsubseteq \dot{\tau}_2$. Fact: $\text{Der}(\dot{T})$ is a regular tree. Given names \dot{T}_1, \dot{T}_2 for regular trees and a name \dot{c} such that $1 \Vdash_{\mathbb{P}} [\dot{c} : \dot{T}_1 \otimes \dot{T}_2 \rightarrow \check{\sigma}]$, let $c : \text{Der}(\dot{T}_1) \otimes \text{Der}(\dot{T}_2) \rightarrow \mathbb{P} \times \sigma$ be any coloring such that for each $r = \langle (\dot{\tau}_1, \alpha), (\dot{\tau}_2, \alpha) \rangle$,

$$\text{First}(c(r)) \Vdash_{\mathbb{P}} \dot{c}(\dot{\tau}_1, \dot{\tau}_2) = \text{Second}(c(r)).$$

SDHL at a not weakly compact cardinal








Corollary

If GCH holds and κ is $(\kappa + d)$ -strong, then there is a forcing extension in which $(\forall \sigma < \kappa)$ SDHL(d, σ, κ) holds, but κ is not weakly compact.

Proof: First force over V to get a model $V[G_1]$ in which SDHL holds at κ , which is also measurable. By a theorem of Hampkins, any non-trivial forcing of size $< \kappa$ followed by a non-trivial $< \kappa$ -closed forcing will make κ NOT weakly compact. Perform any non-trivial forcing of size $< \kappa$ over $V[G_1]$ to get $V[G_1][G_2]$. This will preserve SDHL at κ by the previous theorem. Since κ is measurable in $V[G_1][G_2]$, SDHL holds on a stationary (in fact, measure one) subset of κ . Now perform any non-trivial $< \kappa$ -closed forcing over $V[G_1][G_2]$ to get $V[G_1][G_2][G_3]$. Stationary subsets of κ are preserved, so inside $V[G_1][G_2][G_3]$, SDHL holds on a stationary subset of κ . Thus, SDHL holds at κ in this model.

Question: does SDHL have any large cardinal strength (beyond that of a strongly inaccessible, which is needed for the definition)?

References

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Thank You!