The Halpern-Läuchli Theorem and an Indestructible Partition Relation

Dan Hathaway, joint with Natasha Dobrinen

University of Denver

Daniel.Hathaway@uvm.edu

October 16, 2018

Classical partition relation

Terminology: inaccessible means strongly inaccessible.

Given any $k < \omega$ and any coloring $c : [\mathbb{Q}]^2 \to k$, there is an $X \subseteq \mathbb{Q}$ order isomorphic to $\mathbb{Q}$ such that c uses ≤ 2 colors on $[X]^2$. That is,

$$\mathbb{Q} \to [\mathbb{Q}]^2_{<\omega,2}.$$

This is due to Laver.

The κ-rationals

Definition

Let κ satisfy $\kappa^{<\kappa} = \kappa$. The set of κ-rationals, written $\mathbb{Q}_\kappa$, is the unique κ-saturated linear order of size κ.

Question

In ZFC can we prove

$$\mathbb{Q}_\kappa \to [\mathbb{Q}_\kappa]^2_{<\kappa,<\omega}$$

for every inaccessible κ?

Answer: no!

The classical result does not directly generalize

Why not?

Theorem (Hajnal and Komjáth [6])

Assume there are no Suslin trees of height ω_1. Then after performing Cohen forcing, there is a linear ordering θ of size ω_1 such that for any linear ordering Ω, there is a coloring $c : [\Omega]^2 \to \omega$ such that every subset of Ω order isomorphic to θ does not omit any color.

Corollary

There is a forcing $\mathbb{K}$, of size smaller than the first inaccessible cardinal, such that after forcing with $\mathbb{K}$, every inaccessible cardinal κ satisfies

$$\mathbb{Q}_\kappa \not\to [\mathbb{Q}_\kappa]^2_{\omega,<\omega}.$$

$\mathbb{K}$ just needs to force MA(ω_1), and then add a Cohen real.

Rectangles not squares (a polarized partition relation)

Instead of asking for a set X such that the pairs $p \in [X]^2$ use few colors, we could ask for sets A, B such that the pairs $p \in A \times B$ use few colors.

$[X]^2$ is a "square" and $A \times B$ is a "rectangle".

Fact: given any $k < \omega$ and given any coloring $c : [\mathbb{Q}]^2 \to k$, there are sets $A, B \subseteq \mathbb{Q}$ order isomorphic to $\mathbb{Q}$ such that c uses 1 color on $A \times B$. That is,

$$\binom{\mathbb{Q}}{\mathbb{Q}} \to \binom{\mathbb{Q}}{\mathbb{Q}}^{1,1}_{<\omega,1}$$

Uncountable rectangle partition relation

Let κ be an inaccessible cardinal. Let $(*)_\kappa$ be the partition relation

$$\binom{\mathbb{Q}_\kappa}{\mathbb{Q}_\kappa} \to \binom{\mathbb{Q}_\kappa}{\mathbb{Q}_\kappa}^{1,1}_{<\kappa,2!}$$

Notice we are saying we can cut down to 2 colors, not 1. This is the best possible when $\kappa > \omega$ (see [10]).

$\mathrm{HL}^{tc}(2, {<}\kappa, \kappa)$ is a certain Ramsey theoretic statement.

Theorem (Zhang [10])

Let κ be inaccessible and assume $\mathrm{HL}^{tc}(2, {<}\kappa, \kappa)$ holds. Then $(*)_\kappa$ holds.

Theorem (Zhang [10])

Assume that κ is measurable in the forcing extension to add $(2^\kappa)^+$ many Cohen subsets of κ. Then $\mathrm{HL}^{tc}(2, {<}\kappa, \kappa)$ holds (in the ground model).

Main theorem

Here is our main contribution:

Theorem (Dobrinen and H. [2])

Let κ be inaccessible. Assume $\mathsf{HL}^{tc}(2, {<}\kappa, \kappa)$ holds. Then it still holds after performing any forcing of size $< \kappa$.

Corollary

There is a model of ZFC with an inaccessible cardinal κ such that $(*)_\kappa$ is true after performing any forcing of size $< \kappa$.

So the rectangle partition relation $(*)_\kappa$ can be made indestructible with respect to small forcings, as opposed to the square version which cannot (by the Hajnal and Komjáth result).

HL^{tc}, HL, and SDHL

HL^{tc} has two simpler relatives: HL and SDHL.

We will not define $\text{HL}^{tc}(2, {<}\kappa, \kappa)$ (see [10] for a definition).

For an inaccessible κ, $\text{HL}^{tc}(2, {<}\kappa, \kappa)$ implies $(\forall \sigma < \kappa)$ $\text{HL}(2, \sigma, \kappa)$.

For an inaccessible κ and any nonzero $\sigma < \kappa$, $\text{HL}(2, \sigma, \kappa)$ is equivalent to $\text{SDHL}(2, \sigma, \kappa)$.

We will define $\text{SDHL}(2, \sigma, \kappa)$ on the next slide.

The proof that "$\text{HL}^{tc}(2, {<}\kappa, \kappa)$" cannot be broken by a small forcing is similar to the proof that "$(\forall \sigma < \kappa)$ $\text{SDHL}(2, \sigma, \kappa)$" cannot be broken by a small forcing, just with extra complications. We will prove the SDHL version to illustrate the method.

A word about proving HL

HL(d, σ, ω) can be proved by induction on $d < \omega$ (see [9]). The successor step involves a fusion argument. This cannot be generalized to the $\kappa > \omega$ case because the intersection of a decreasing sequence of regular trees may not be regular.

There is another proof of HL(d, σ, ω) (see [3]) which adds many Cohen reals by forcing, and uses an ultrafilter in the extension to make selections. This generalizes to the $\kappa > \omega$ case if we assume that κ is measurable in the extension:

Theorem (Dobrinen and H. [1])

Let $\lambda > \kappa$ satisfy $\lambda \to (\kappa)^d_\kappa$. Assume κ is measurable in the forcing extension where we add λ many Cohen subsets of κ. Then HL(d, σ, κ) holds (in the ground model).

(Woodin, see [4] for a proof): if GCH holds and κ is $(\kappa + d)$-strong, then there is a forcing extension in which κ is measurable and remains measurable after adding $\lambda = \kappa^{+d}$ Cohen subsets of κ.

Some definitions

Definition

Let κ be a cardinal. A tree $T \subseteq {}^{<\kappa}\kappa$ is **regular** iff

1) it is perfect,
2) every maximal branch has length κ, and
3) it is a κ-tree (every level $T(\alpha) := T \cap {}^{\alpha}\kappa$ of T has size $< \kappa$).

Note: If κ is regular and there is a regular κ-tree, then κ is inaccessible.

Definition

Given sets $T_0, T_1 \subseteq {}^{<\kappa}\kappa$, the set $T_0 \otimes T_1$ consists of all the pairs $\langle t_0, t_1 \rangle \in T_0 \times T_1$ such that t_0 and t_1 have the same length.

Definition

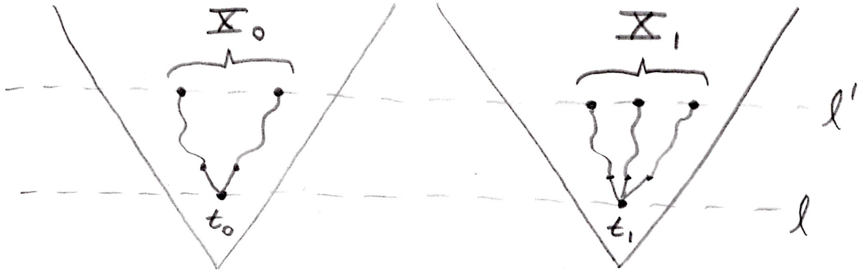
Given sets $A, D \subseteq {}^{<\kappa}\kappa$, we say that D **dominates** A iff each $a \in A$ is extended by some $d \in D$.

A definition of SDHL

SDHL stands for "Somewhere Dense Halpern-Läuchli".

Definition

Let $0 < \sigma < \kappa$ be cardinals. SDHL$(2, \sigma, \kappa)$ is the statement that given any regular trees $T_0, T_1 \subseteq {}^{<\kappa}\kappa$ and any coloring $c : T_0 \otimes T_1 \to \sigma$, there are levels $l < l' < \kappa$, a sequence of nodes $\langle t_i \in T_i(l) : i < 2 \rangle$, and a sequence of sets $\langle X_i \subseteq T_i(l') : i < 2 \rangle$ such that each X_i dominates $\text{Succ}_{T_i}(t_i)$ and c is constant on $X_0 \otimes X_1$.



SDHL version of main theorem: part 1

Theorem 1 (Dobrinen and H. [2])

Let $0 < \sigma < \kappa$. Let $\mathbb{P}$ be a forcing of size $< \kappa$. Then $\mathrm{SDHL}(2, \sigma \cdot |\mathbb{P}|, \kappa)$ implies $1 \Vdash_{\mathbb{P}} \mathrm{SDHL}(2, \sigma, \kappa)$.

Given a name $\dot{T}$ for a regular tree, let $\mathrm{Der}(\dot{T})$ be the set of all equivalence classes of pairs $(\dot{\tau}, \alpha)$ such that

$$1 \Vdash_{\mathbb{P}} (\dot{\tau} \in \dot{T} \text{ and } \mathrm{Length}(\dot{\tau}) = \check{\alpha}),$$

where $(\dot{\tau}_1, \alpha_1) \cong (\dot{\tau}_2, \alpha_2)$ iff $1 \Vdash_{\mathbb{P}} (\dot{\tau}_1 = \dot{\tau}_2)$. Order $\mathrm{Der}(\dot{T})$ by $[(\dot{\tau}_1, \alpha_1)] \leq [(\dot{\tau}_2, \alpha_2)]$ iff $1 \Vdash_{\mathbb{P}} \dot{\tau}_1 \sqsubseteq \dot{\tau}_2$. Given $X \subseteq \mathrm{Der}(\dot{T})$, let $\mathrm{Names}(X)$ be the set of names that occur in the elements of X.

Crucial Fact (the "Derived Tree Theorem"): $\mathrm{Der}(\dot{T})$ is a regular tree. Also, given any $[(\dot{\tau}, \alpha)] \in \mathrm{Der}(\dot{T})$, the successors of the node named by $\dot{\tau}$ are all named by successors of $[(\dot{\tau}, \alpha)]$ in $\mathrm{Der}(\dot{T})$.

SDHL version of main theorem: part 2

Let $\dot{T}_0, \dot{T}_1$ be names for regular trees and let $\dot{c}$ be a name such that

$$1 \Vdash_{\mathbb{P}} [\dot{c} : \dot{T}_1 \otimes \dot{T}_2 \to \check{\sigma}].$$

Let $d : \mathrm{Der}(\dot{T}_0) \otimes \mathrm{Der}(\dot{T}_1) \to \mathbb{P} \times \sigma$ be any coloring such that for each $r = \langle [(\dot{\tau}_0, \alpha)], [(\dot{\tau}_1, \alpha)] \rangle$,

$$\mathrm{First}(d(r)) \Vdash_{\mathbb{P}} \dot{c}(\dot{\tau}_0, \dot{\tau}_1) = \mathrm{Second}(d(r)).$$

Apply $\mathrm{SDHL}(d, |\mathbb{P}| \cdot \sigma, \kappa)$ to get $l < l' < \kappa$, $X_0 \subseteq \mathrm{Der}(\dot{T}_0)(l')$, $X_1 \subseteq \mathrm{Der}(\dot{T}_1)(l')$, and nodes $t_0 \in \mathrm{Der}(\dot{T}_0)(l)$ and $t_1 \in \mathrm{Der}(\dot{T}_1)(l)$ such that X_i dominates the successors of t_i (for $i = 0, 1$) and d is monochromatic on $X_0 \otimes X_1$, say with color (p, δ). By the Derived Tree Theorem, p forces that $\dot{c}$ is monochromatic on $\mathrm{Names}(X_0) \otimes \mathrm{Names}(X_1)$, with color $\check{\delta}$. Also, $\mathrm{Names}(X_i)$ dominates the successors of the node named by t_i (for $i = 0, 1$). □

$<\kappa$-closed forcings

So now we know that SDHL, HL, and HL^{tc} cannot be broken by forcings of size $< \kappa$.

Here is another preservation theorem (but it does not hold for HL^{tc}):

Theorem 2 (Dobrinen and H. [2])

Suppose κ is measurable and $0 < \sigma < \kappa$. Let $\mathbb{P}$ be a $<\kappa$-closed forcing. Then $\mathrm{SDHL}(2, \sigma, \kappa)$ implies $1 \Vdash_{\mathbb{P}} \mathrm{SDHL}(2, \sigma, \kappa)$.

Ingredients in the proof (2 and 3 are in the next two slides):

1) a $<\kappa$-closed forcing will preserve stationary subsets of κ.
2) if SDHL holds at a measurable cardinal, it holds on a measure one (and therefore stationary) set below the cardinal.
3) if SDHL holds on a stationary set below a cardinal, it holds at the cardinal.

Downward (and upward) reflection at a measurable

$\mathsf{SDHL}(2, \sigma, \kappa)$ is a statement about $V_{\kappa+1}$.

The following proposition also works for either HL or HL^{tc} in place of SDHL.

Proposition (Dobrinen and H. [2])

Let κ be a measurable cardinal with a normal measure $\mathcal{U}$. Fix $0 < \sigma < \kappa$. Then $\mathsf{SDHL}(2, \sigma, \kappa)$ iff

$$\{\alpha < \kappa : \mathsf{SDHL}(2, \sigma, \alpha)\} \in \mathcal{U}.$$

Proof: Let $j : V \to M$ be the ultrapower embedding.

Because $V_{\kappa+1} \subseteq M$, $\mathsf{SDHL}(2, \sigma, \kappa) \Leftrightarrow \mathsf{SDHL}(2, \sigma, \kappa)^M$.

By Łos's Theorem, $\mathsf{SDHL}(2, \sigma, \kappa)^M \Leftrightarrow \{\alpha < \kappa : \mathsf{SDHL}(2, \sigma, \alpha)\} \in \mathcal{U}$.

Upward stationary reflection

The following is not true for HL^{tc}.

Proposition (Dobrinen and H. [2])

Let κ be a cardinal. Assume either

1) κ is inaccessible or
2) $\mathrm{cf}(\kappa) > \omega$ and κ is the limit of inaccessible cardinals.

Assume that

$$S := \{\alpha < \kappa : \mathrm{SDHL}(2, \sigma, \alpha)\}$$

is stationary. Then $\mathrm{SDHL}(2, \sigma, \kappa)$ holds.

Proof: Let $\langle T_i \subseteq {}^{<\kappa}\kappa : i < 2 \rangle$ be a sequence of regular trees and let $c : \bigotimes_{i<2} T_i \to \sigma$ be a coloring. If we can find an $\alpha < \kappa$ such that each $T_i \cap {}^{<\alpha}\kappa$ is a regular α-tree and $\mathrm{SDHL}(2, \sigma, \alpha)$ holds, then we will be done. An elementary argument shows that for each $i < 2$, there is a club $C_i \subseteq \kappa$ such that $(\forall \alpha \in C_i)\ T_i \cap {}^{<\alpha}\kappa$ is a regular α-tree. The set $\bigcap_{i<2} C_i$ is a club, so it must intersect S. An $\alpha < \kappa$ in the intersection is as desired.

SDHL at a not weakly compact cardinal

Let Ψ be the statement $(\forall\sigma < \kappa)\,\mathrm{SDHL}(2, \sigma, \kappa)$.

Corollary (Dobrinen and H. [2])

Let κ be measurable and assume Ψ holds. Then after performing any non-trivial forcing of size $< \kappa$ followed by a non-trivial $<\kappa$-closed forcing, Ψ will still hold but κ will not be weakly compact.

Proof: By a theorem of Hamkins [7], any non-trivial forcing of size $< \kappa$ followed by a non-trivial $<\kappa$-closed forcing will make κ NOT weakly compact.

Perform any non-trivial forcing of size $< \kappa$ over V to get $V[G_1]$. This will preserve Ψ by Theorem 1. Now perform any non-trivial $<\kappa$-closed forcing over $V[G_1]$ to get $V[G_1][G_2]$. Since κ is measurable in $V[G_1]$, by Theorem 2 we have that Ψ holds in $V[G_1][G_2]$.

Large cardinal strength?

Zhang [10] has independently shown that SDHL can hold at a cardinal that is not weakly compact.

On the other hand, Zhang [10] has shown that $\mathrm{HL}^{tc}(2, {<}\kappa, \kappa)$ implies that κ is weakly compact.

Question

Can SDHL ever fail?

In pacticular, does SDHL have any large cardinal strength?

Also, we must ask the following:

Question

Can $(*)_\kappa$ ever fail?

References

[1] N. Dobrinen and D. Hathaway. *The Halpern-Läuchli theorem at a measurable cardinal*. The Journal of Symbolic Logic 82 (2017), 17-33.

[2] N. Dobrinen and D. Hathaway. *Forcing and the Halpern-Läuchli theorem*. To appear.

[3] I. Farah and S. Todorcevic. *Some applications of the method of forcing*. Moscow: Yenisei, 1995.

[4] S. Friedman and K. Thompson. *Perfect trees and elementary embeddings*. The Journal of Symbolic Logic 73 (2008).

[5] J. D. Halpern and H. Lauchli. *A partition theorem*. Transactions of the American Mathematical Society 124 (1966).

[6] A. Hajnal and P. Komjaťh. *A strongly non-Ramsey order type*. Combinatorica 17 (1997).

[7] J. D. Hamkins. *Small forcing makes any cardinal superdestructible*. Journal of Symbolic Logic 63 (1998), 51-58.

[8] S. Shelah. *Strong partition relations below the power set: consistency - was Sierpinski right? II*. Sets, Graphs and Numbers 60 (1991).

[9] S. Todorcevic. *Introduction to Ramsey spaces*. Princeton, NJ: Princeton University Press, 2010.

[10] J. Zhang. *A tail cone version of the Halpern-Läuchli Theorem at a large cardinal*. To appear.

Thank You!