# Ramsey Theory on Generalized Baire Space

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# Ramsey Sets

Let $\kappa$ be an infinite cardinal.

**Definition**

A set $\mathcal{X} \subseteq [\kappa]^\kappa$ is **Ramsey** iff there is some $H \in [\kappa]^\kappa$ such that either

- every $X \in [H]^\kappa$ is in $\mathcal{X}$ or
- no $X \in [H]^\kappa$ is in $\mathcal{X}$.

$H$ is called **homogeneous** for $\mathcal{X}$.

# Generalized Baire Space

Given an ordinal $\alpha < \kappa$ and a set $A \subseteq \alpha$, let

$$\mathcal{B}_{A,\alpha} := \{X \in [\kappa]^\kappa : X \cap \alpha = A\}.$$

The $\mathcal{B}_{A,\alpha}$'s form a basis for the *standard topology on generalized Baire space*. The topology is too fine, because if $\kappa > \omega$, there is a clopen set $\mathcal{X} \in [\kappa]^\kappa$ that is not Ramsey (ZFC): using a coloring $c : [\kappa]^\omega \to 2$ with no $H \in [\kappa]^\kappa$ satisfying $|c``[H]^\omega| = 1$, let $\mathcal{X} \subseteq [\kappa]^\kappa$ be the set of all $X$ whose first $\omega$ elements $X'$ satisfy $c(X') = 1$.

How to get a coarser topology? Use the topology generated by sets of the form

$$\text{Match}(A, B) := \{X \in [\kappa]^\kappa : X \cap (A \cup B) = A\}$$

where $A$ and $B$ must be *small*.

# Patterns

## Definition

A **pattern** is a pair $(A, B)$ such that $A, B \subseteq \kappa$ and $A \cap B = \emptyset$. A set $X \in [\kappa]^\kappa$ **matches** $(A, B)$ iff $X \cap (A \cup B) = A$. That is, $A \subseteq X$ and $B \cap X = \emptyset$. Match$(A, B)$ is the set of $X \in [\kappa]^\kappa$ that match $(A, B)$.

## Definition

Given $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\kappa)$, $(A, B)$ is an $(\mathcal{A}, \mathcal{B})$-pattern iff $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

## Definition

$\Sigma(\mathcal{A}, \mathcal{B})$ is the collection of all sets $\mathcal{X}$ of the form

$$\mathcal{X}_\mathcal{Q} := \{X \in [\kappa]^\kappa : X \text{ matches some } (A, B) \in \mathcal{Q}\}$$

for some set $\mathcal{Q}$ of $(\mathcal{A}, \mathcal{B})$-patterns.
$\Delta(\mathcal{A}, \mathcal{B}) = \{\mathcal{X} : \mathcal{X}, [\kappa]^\kappa - \mathcal{X} \in \Sigma(\mathcal{A}, \mathcal{B})\}$.

# Examples

$\Sigma([\kappa]^{<\kappa}, [\kappa]^{<\kappa})$ is the collection of all open subsets of generalized Baire space, and $\Delta([\kappa]^{<\kappa}, [\kappa]^{<\kappa})$ is the collection of all clopen subsets. If $\kappa > \omega$, some set in $\Delta([\kappa]^{<\kappa}, [\kappa]^{<\kappa})$ is not Ramsey.

When $\mathcal{A}$ or $\mathcal{B}$ is enlarged, $\Sigma(\mathcal{A}, \mathcal{B})$ becomes finer.

Silver: every Analytic set in the topology $\Sigma([\omega]^{<\omega}, [\omega]^{<\omega})$ is Ramsey.

Ellentuck: every Analytic set in the topology $\Sigma([\omega]^{<\omega}, [\omega]^{\leq\omega})$ is Ramsey.

Large cardinals imply that every $\mathcal{X} \subseteq [\omega]^{\omega}$ in $L(\mathbb{R})$ is Ramsey.

# $|A|, |B|$ bounded below $\kappa$

**Theorem**

Fix $\gamma < \kappa$. Every $\Delta([\kappa]^{<\gamma}, [\kappa]^{<\gamma})$ set is Ramsey.

**Open**

Fix $\gamma < \kappa$. Is every $\Sigma([\kappa]^{<\gamma}, [\kappa]^{<\gamma})$ set Ramsey?

- Is every $\Sigma([\omega_1]^2, [\omega_1]^1)$ set Ramsey?
- Is every $\Sigma([\kappa]^\omega, [\kappa]^1)$ set Ramsey if $\kappa$ is measurable?

# $|A| = 2.$

Assume the Axiom of Choice.

## Theorem

The following are equivalent:

- $\kappa$ is weakly compact,
- every $\Delta([\kappa]^2, [\kappa]^{<\kappa})$ set is Ramsey,
- every $\Sigma([\kappa]^2, [\kappa]^{<\kappa})$ set is Ramsey,
- $(\forall n \in \omega)$ every $\Sigma([\kappa]^n, [\kappa]^{<\kappa})$ set is Ramsey.

# $|A| < \omega$

Using a similar argument:

**Theorem**

If $\kappa$ is Ramsey cardinal, then every $\Sigma([\kappa]^{<\omega}, [\kappa]^{<\kappa})$ set is Ramsey.

**Question**

If every $\Sigma([\kappa]^{<\omega}, [\kappa]^{<\kappa})$ set is Ramsey, what kind of large cardinal is $\kappa$?

If $\kappa$ is measurable, $B$ can have size $\kappa$:

**Theorem**

Let $\mathcal{U}$ be a $\kappa$-complete ultrafilter on $\kappa$. Then every $\Sigma([\kappa]^{<\omega}, \mathcal{P}(\kappa) - \mathcal{U})$ set is Ramsey.

# When $\mathcal{Q} \subseteq L$ or $\mathcal{Q} \in L$

**Theorem**

Let $\kappa > \omega$ be a Ramsey cardinal. Let $\mathcal{Q} \subseteq L$ be a set of patterns. The set $\mathcal{X}_{\mathcal{Q}} \subseteq [\kappa]^\kappa$ generated by $\mathcal{Q}$ (in $V$) is Ramsey.

**Theorem**

Assume $0^\#$ exists. Let $\kappa > \omega$ be a cardinal. Let $\mathcal{Q} \in L$ be a set of patterns. The set $\mathcal{X}_{\mathcal{Q}} \subseteq [\kappa]^\kappa$ generated by $\mathcal{Q}$ (in $V$) is Ramsey.

**Question**

Does it follow from large cardinals, or is it even consistent with $\mathrm{ZFC}$, that for every set $\mathcal{Q} \in L(\mathbb{R})$ of $([\omega_1]^{<\omega_1}, [\omega_1]^{<\omega_1})$-patterns, the set $\mathcal{X}_{\mathcal{Q}} \subseteq [\omega_1]^{\omega_1}$ generated by $\mathcal{Q}$ (in $V$) is Ramsey?

# Thank You!

# References


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