Generic Coding with Help

Dan Hathaway

University of Denver

Daniel.Hathaway@du.edu

April 18, 2018

Motivation

Some reals ($0^\#$, etc) are not in any forcing extension of L. This persists even when forcing over V: whenever H is V-generic, in $V[H]$ there is no G generic over L such that $0^\# \in L[G]$.

However, there are reals c_1, c_2 both Cohen generic over L such that $0^\# \in L[c_1, c_2]$. It must be that c_2 is not generic over $L[c_1]$ and c_1 is not generic over $L[c_2]$.

Suppose a is an arbitrary real **not in L**. Is there a generic G over L such that $0^\# \in L[a, G]$? We will see that the answer is YES!

$\mathcal{P}(\omega)$ Theorem

Theorem

Let M be a transitive model of ZF and assume $|\mathcal{P}(\mathbb{R})^M| = \omega$. Let $a \in \mathcal{P}(\omega) - M$. Then for any $x \in \mathcal{P}(\omega)$, there is some G such that

1) G is $\mathbb{H}_{\omega,\omega}$-generic over M and
2) $x \in L[a, G]$.

- We will define $\mathbb{H}_{\omega,\omega}$ soon.
- G need not be generic over $L[a]$ (we could have $x = a^\#$).
- By universality of the collapsing poset, we can replace $\mathbb{H}_{\omega,\omega}$ with $\text{Col}(\omega, \delta)$ where $\delta = (2^\omega)^M$.

Generic generics over L

Every set is generically generic over L, with help from $0^\#$.

Corollary

Let λ be a cardinal and $X \subseteq \lambda$. Whenever $V[H]$ is a forcing extension of V in which λ is countable, there is some $G \in V[H]$ such that

1) G is $\mathrm{Col}(\omega, \lambda) * \mathbb{H}_{\omega,\omega}$-generic over L and
2) $X \in L[0^\#, G]$.

In $V[H]$, we first build G_1 that is $\mathrm{Col}(\omega, \lambda)$-generic over L. Note that $0^\# \notin L[G_1]$. From G_1 we can recover a surjection $s : \omega \to \lambda$. Fix an $x \in \mathcal{P}(\omega)$ such that $X \in L[s, x]$. Then we build G_2 that is $\mathbb{H}_{\omega,\omega}^{L[G_1]}$-generic over $L[G_1]$ such that $x \in L[0^\#, G_2]$.

$0^\#$ can be replaced by any real in $V[H]$ that is not generic over L.

${}^{\omega}\lambda$ Theorem

There is a more direct way to get that every $x \in {}^{\omega}\lambda$ is generically generic with help:

Theorem

Let λ be a singular cardinal of cofinality ω. Let M be a transitive model of ZFC such that $\lambda \in M$ and ${}^{<\lambda}2 \subseteq M$. Let $\delta = (2^{\lambda})^{M}$. Let $A \in \mathcal{P}(\lambda) - M$. Then in any forcing extension $V[H]$ of V in which $|\mathcal{P}^{M}(\delta)| = \omega$, for any $x \in {}^{\omega}\lambda$ in $V[H]$, there is some G such that

1) G is $\mathbb{H}_{\omega,\lambda}$-generic over M and
2) $x \in M[A, G]$.

- We will define $\mathbb{H}_{\omega,\lambda}$ soon.
- We work in the extension $V[H]$ because in V, the set $\mathcal{P}(2^{\lambda})^{M}$ must be uncountable.
- It is important that $A \in V$.
- We can replace $\mathbb{H}_{\omega,\lambda}$ with $\mathrm{Col}(\omega, \delta)$.

Proof of theorems: Part 1

Definition

Let λ be an infinite cardinal. The forcing $\mathbb{H}_{\omega,\lambda}$ consists of all trees $T \subseteq {}^{<\omega}\lambda$ such that for each $t \sqsupseteq \text{Stem}(T)$, $\{\gamma : t^\frown\gamma \notin T\}$ has size $< \lambda$. The ordering is by inclusion.

$\mathbb{H}_{\omega,\lambda}$ is a variant of Hechler forcing. It adds a "fast growing" function from ω to λ. Note: $\mathbb{H}_{\omega,\lambda}$ has a dense subset of size $\lambda^\omega \leq 2^\lambda$.

Definition

Fix $A \subseteq \lambda$. Given $T_1, T_2 \in \mathbb{H}_{\omega,\lambda}$, we write $T_2 \leq^A T_1$ iff $T_2 \leq T_1$ and letting $t_2 = \text{Stem}(T_2)$ and $t_1 = \text{Stem}(T_1)$,

$$(\forall n \in \mathrm{Dom}(t_2) - \mathrm{Dom}(t_1))\, t_2(n) \notin A.$$

$T_2 \leq^A T_1$ implies "T_2 does not hit A more than T_1 already does".

Proof of theorems: Part 2

Definition

Given $T \in \mathbb{H}_{\omega,\lambda}$ and $t \in T$, we write $t' \sqsupseteq_T t$ iff $t' \sqsupseteq t$ and $t' \in T$. Given $A \subseteq \lambda$, we write $t' \sqsupseteq_T^A t$ iff $t' \sqsupseteq_T t$ and

$$(\forall n \in \mathrm{Dom}(t') - \mathrm{Dom}(t))\, t'(n) \notin A.$$

Definition

A set $S \subseteq {}^{<\omega}\lambda$ is **large** iff given any $T \in \mathbb{H}_{\omega,\lambda}$, there is some $t' \sqsupseteq_T \mathrm{Stem}(T)$ such that $t' \in S$.

Large means "the set of stems of some dense set":

Lemma

$S \subseteq {}^{<\omega}\lambda$ is large iff there is some dense $D \subseteq \mathbb{H}_{\omega,\lambda}$ such that

$$S = \{t \in {}^{<\omega}\lambda : (\exists T \in D)\, t = Stem(T)\}.$$

Proof of theorems: Part 3

Fix $A \subseteq \lambda$ and a function $\eta : A \to \lambda$ such that $(\forall \beta < \lambda)\, \eta^{-1}(\beta)$ has size λ.

Let G be $\mathbb{H}_{\omega,\lambda}$-generic over some model M. Let $g = \cap G$. Then $g : \omega \to \lambda$.

Idea: whenever $g(n) \in A$, the value of $\eta(g(n))$ is a single piece of information that has been encoded. Given $x \in {}^{\omega}\lambda$ and letting $n_0 < n_1 < \ldots$ be the n's such that $g(n) \in A$, we hope to encode x as

$$x = \langle \eta(g(n_i)) : i < \omega \rangle.$$

Issue: does the requirement that G be generic over M cause there to be unwanted n's such that $g(n) \in A$? If we can hit dense sets by making $\leq^A$ extensions, then we can build a G which hits all dense sets in M without interfering with our encoding.

Proof of theorems: Part 4

How do we hit dense subsets of $\mathbb{H}_{\omega,\lambda}$ by making $\leq^A$-extensions?

Ingredient:

Sticking Out Observation

Let M be a transitive model such that ${}^{<\lambda}2 \subseteq M$. Let $A \in [\lambda]^\lambda$ and assume $(\forall B \in [A]^\lambda)\, B \notin M$. Then if $B \in [\lambda]^\lambda \cap M$, then $B - A$ has size λ.

Proof: If $|B - A| < \lambda$, then $B - A \in M$ therefore $B \cap A \in M$ and $B \cap A$ is a size λ subset of A.

Thus, given $T \in \mathbb{H}_{\omega,\lambda}$, if we need to extend the stem t of T by one and we are given λ choices $t^\frown\gamma$ for how to do this, then λ of the γ will not be in A.

Proof of theorems: Part 5

Definition

Let $S \subseteq {}^{<\omega}\lambda$ and $t \in {}^{<\omega}\lambda$.

- t is 0-S-**reachable** iff $t \in S$.
- for $\alpha > 0$, t is α-S-**reachable** iff

$$\{\gamma < \lambda : (\exists \beta < \alpha)\, t^\frown\gamma \text{ is } \beta\text{-}S\text{-reachable}\}$$

  has size λ.

Lemma

Let $D \subseteq \mathbb{H}_{\omega,\lambda}$ be dense. Let $S \subseteq {}^{<\omega}\lambda$ be the set

$$S = \{t : (\exists T \in D)\, t = \mathrm{Stem}(T)\}.$$

Then $(\forall t \in {}^{<\omega}\lambda)(\exists \alpha \in \mathrm{Ord})\, t$ is α-S-reachable.

Proof of theorems: Part 6

Note: this proof does *not* work for $\mathbb{H}_{\kappa,\lambda}$ for $\kappa > \omega$.

Main Lemma

Let M be a transitive model such that ${}^{<\lambda}2 \subseteq M$. Let $A \in [\lambda]^\lambda$ be such that $(\forall B \in [A]^\lambda)\, B \notin M$. Let $S \subseteq {}^{<\omega}\lambda$ be large (in M). Let $T \in \mathbb{H}^M_{\omega,\lambda}$ and $t = \text{Stem}(T)$. Then there exists some $t' \sqsupseteq^A_T t$ in S.

Proof: We have that t is α-S-reachable for some α.

If $\alpha = 0$, then $t \in S$ and we are done.

If $\alpha > 0$, then consider $B = \{\gamma < \lambda : (\exists \beta < \alpha)\, t^\frown\gamma \text{ is } \beta\text{-}S\text{-reachable}\}$. It is in M and has size λ, so by the "sticking out observation", $B - A$ has size λ. Thus, there is some $\gamma_0 \in B - A$ and such that $t^\frown\gamma_0 \in T$. If $t^\frown\gamma_0$ is 0-S-reachable we are done. Otherwise we can find some $t^\frown\gamma_0^\frown\gamma_1$, etc. This process will terminate after finitely many stages.

Proof of theorems: Part 7

Corollary

Let M be a transitive model such that ${}^{<\lambda}2 \subseteq M$. Let $A \in [\lambda]^\lambda$ be such that $(\forall B \in [A]^\lambda)\, B \notin M$. Let $D \in \mathcal{P}^M(\mathbb{H}^M_{\omega,\lambda})$ be open dense (in M). Let $T \in \mathbb{H}^M_{\omega,\lambda}$. Then there exists some $T' \leq^A T$ in D.

Thus we can do a construction (in V) to hit the dense^M subsets of $\mathbb{H}^M_{\omega,\lambda}$ by making only $\leq^A$ extensions.

We can alternate between doing $\leq^A$-extensions to hit the dense sets in M, and doing $\leq$-extensions to encode more and more of some $x \in {}^\omega\lambda$.

Proof of theorems: Part 8

But how to get an $A \in [\lambda]^\lambda$ such that $(\forall B \in [A]^\lambda)\, B \notin M$?

ω-Stuttering Lemma

For every $\tilde{A} \subseteq \omega$, there is some $A =_T \tilde{A}$ that is computable from every infinite subset of itself.

One can generalize this using bijections from ${}^\sigma 2$ to 2^σ for $\sigma < \lambda$.

λ-Stuttering Lemma

Let M be a transitive model of ZFC such that $\lambda \in M$. Suppose $(\forall \sigma < \lambda)\, (2^\sigma)^M \leq \lambda$. For every $\tilde{A} \subseteq \lambda$, there is some $A \in [\lambda]^\lambda$ such that $(\forall B \in [A]^\lambda)\, M[B] = M[\tilde{A}]$.

Another application of the Main Lemma

We have now proved the theorems!

Another application of the main lemma:

Theorem

Assume AD^+. Fix $a \in \mathbb{R}$. There is a Borel function $f_a : {}^\omega\omega \to {}^\omega\omega$ with the following property: given any $g : {}^\omega\omega \to {}^\omega\omega$,

$$g \cap f_a = \emptyset \Rightarrow a \in L[C]$$

where $C \subseteq \mathrm{Ord}$ is any ∞-Borel code for g.

$\mathcal{P}(\omega_1)$ Conjecture

Conjecture

Assume the Axiom of Choice and that there are large cardinals. There is

- an inner model $M \supseteq \mathbb{R}$ satisfying AD,
- a set $A \subseteq \omega_1$, and
- a cardinal μ

such that whenever $X \subseteq \omega_1$ and H is $\mathrm{Col}(\omega_1, \mu)$-generic over V, there is some $G \in V[H]$ such that

1) G is generic over M by a countably closed forcing and
2) $X \in M[A, G]$.

With the right cardinality assumption, we should not need to pass to $V[H]$.

The $\mathbb{H}_{\omega_1,\mathbb{R}}$ game: Part 1

We can try to prove the $\mathcal{P}(\omega_1)$ conjecture using the poset $\mathbb{H}_{\omega_1,\mathbb{R}}$ defined in the natural way (conditions are trees $T \subseteq {}^{<\omega_1}\mathbb{R}$ where each node has all but countably many children).

Given $S \subseteq {}^{<\omega_1}\mathbb{R}$, the length ω_1 game $\mathbb{H}_{\omega_1,\mathbb{R}}(S)$ is as follows:

On round $\alpha < \omega_1$, Player I plays some $C_\alpha \in [\mathbb{R}]^\omega$ and then Player II plays some $r_\alpha \in \mathbb{R} - C_\alpha$. Player II wins the game iff for some $\alpha < \omega_1$,

$$\langle r_\beta : \beta < \alpha \rangle \in S.$$

The game is *closed*.

Player I has a winning strategy iff S is not large.

The $\mathbb{H}_{\omega_1,\mathbb{R}}$ game: Part 2

Suppose $A \subseteq \mathbb{R}$ has size ω_1 and $(\forall B \in [A]^{\omega_1})\, B \notin M$. Assume Player II has a strategy $\Gamma \in M$ for the $\mathbb{H}_{\omega_1,\mathbb{R}}(S)$-game that is a winning strategy in **both** M and V. Assume the Axiom of Choice in V. Then

- Player I can play so that II always responds (using Γ) with a real not in A (there are at least ω_1 responses II would make, so by the sticking out observation ω_1 responses must not be in A).
- The play of the game must terminate at some stage before ω_1, because otherwise I wins.
- It cannot terminate by player II getting stuck because Γ is a winning strategy for II.
- Thus, it must terminate by the sequence constructed so far being an element of S.

Thus, if for each large S there is such a Γ for $\mathbb{H}_{\omega_1,\mathbb{R}}(S)$, then the $\mathcal{P}(\omega_1)$ conjecture is true.

References

J. Baumgartner and P. Dordal. *Adjoining dominating functions*. The Journal of Symbolic Logic 50 (1985).

D. Hathaway. *Disjoint Borel Functions*. Ann. Pure Appl. Logic 168 (2017).

—. *Disjoint Infinity-Borel Functions*. Submitted for Publication.