Generic Coding with Help

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How much information can be forced into a real?

Some reals ($0^\#$, etc) are not in any forcing extension of L. This persists even when forcing over V: whenever H is V-generic, in $V[H]$ there is no G generic over L such that $0^\# \in L[G]$.

However, there are reals c_1, c_2 both Cohen generic over L such that $0^\# \in L[c_1, c_2]$. It must be that c_2 is not generic over $L[c_1]$ and c_1 is not generic over $L[c_2]$.

Suppose a is an arbitrary real **not in L**. Is there a generic G over L such that $0^\# \in L[a, G]$? We will see that the answer is YES!

Encoding any real into two Cohen reals (Mostowski)

Let M be a c.t.m. of ZFC. Fix $x \in {}^{\omega}2$. We will find $g, h \in {}^{\omega}2$ Cohen generic over M such that $x = g$ XOR h.

Let $\langle D_n : n < \omega \rangle$ be an enumeration of all dense subsets (in M) of Cohen forcing.

Let $g_0 \in {}^{<\omega}2$ be in D_0. Let $h_0 \in {}^{<\omega}2$ be the same length as g_0 such that

$$g_0 \text{ XOR } h_0 = x \restriction |g_0|.$$

Let $h_1 \sqsupseteq h_0$ be in D_0. Let $g_1 \sqsupseteq g_0$ be the same length as h_1 such that

$$g_1 \text{ XOR } h_1 = x \restriction |g_1|.$$

Let $g_2 \sqsupseteq g_1$ be in D_1. Let $h_2 \sqsupseteq h_1$ be the same length as g_2 such that

$$g_2 \text{ XOR } h_2 = x \restriction |g_2|.$$

etc. The reals $g = \bigcup g_i$ and $h = \bigcup h_i$ are as desired.

More complicated patterns with Cohen reals

Theorem (Mostowski)

Let M be a c.t.m. of ZFC. Fix $l \in \omega$. Let $\mathcal{A}$ be a collection of subsets of $\{0, ..., l\}$ that contains all singletons and is closed under subsets. Fix any $x \in {}^{\omega}2$. Then there are reals $g_0, ..., g_l$ all Cohen generic over M such that for any $A \subseteq \{0, ..., l\}$,

1) if $A \in \mathcal{A}$, then $\{g_i : i \in A\}$ is contained in a forcing extension of M;
2) if $A \notin \mathcal{A}$, then $x \in L[\{g_i : i \in A\}]$.

In their paper Set Theoretic Blockchains [1], the authors prove more complicated versions of this.

The main theorem

Let M be a c.t.m. of ZFC. Say that $a \in {}^\omega 2$ is **helpful** iff for any $x \in {}^\omega 2$, there is a G generic over M such that $x \in L[a, G]$.

Theorem (Habič et al [1])

If a is Cohen generic over M, it is helpful.

Theorem (Habič, Sy Friedman)

If a is unbounded over M, it is helpful.

Theorem (Sy Friedman)

If a is Sacks generic over M, it is helpful.

Our main theorem:

Theorem [4]

If a is *any* real not in M, it is helpful.

Main Theorem: Part 1

Recall *Tree Hechler Forcing*, whose conditions are trees with cofinite splitting beyond their stems:

Definition

The forcing $\mathbb{H}$ consists of the trees $T \subseteq {}^{<\omega}\omega$ such that for all $t \sqsupseteq \text{Stem}(T)$ in T,

$$\{z \in \omega : t^\frown z \notin T\} \text{ is finite.}$$

The ordering is by inclusion.

Given a generic G for $\mathbb{H}$, G can be recovered from $\bigcup \bigcap G$ (the union of the stems of conditions in G), which is a function from ω to ω.

Main Theorem: Part 2

Given a set $A \subseteq \omega$, we can define an auxillary ordering $\leq_A$ on $\mathbb{H}$:

Definition

Let $A \subseteq \omega$. Given $t, t' \in {}^{<\omega}\omega$, we write $t' \sqsupseteq_A t$ iff $t' \sqsupseteq t$ and

$$(\forall n \in \mathrm{Dom}(t') - \mathrm{Dom}(t))\, t'(n) \notin A.$$

So if $t' \sqsupseteq_A t$, then t' does not "hit" A any more than t already does.

Definition

Let $A \subseteq \omega$. Given $T, T' \in \mathbb{H}$, we write $T' \leq_A T$ iff $T' \leq T$ and $\mathrm{Stem}(T') \sqsupseteq_A \mathrm{Stem}(T)$.

So $T' \leq_A T$ means that T' is stronger than T and the stem of T' does not hit A any more than the stem of T.

Main Theorem: Part 3

Given an $\mathbb{H}$-generic G (over a transitive model) and an $A \subseteq \omega$, here is how we can decode a real $x \in {}^{\omega}2$: let $a_0, a_1, \ldots$ be the increasing enumeration of A. Let $\eta_A : A \to 2$ be the function $\eta_A(a_i) = 0$ if i is even, and 1 if i is odd. Consider $g = \bigcup \bigcap G$ (g is the union of the stems of conditions in G). We have $g : \omega \to \omega$. Let

$$n_0 < n_1 < \ldots$$

be the increasing enumeration of the set of $n \in \omega$ such that $g(n) \in A$. We can now "decode" the real $\langle \eta_A(g(n_i)) : i < \omega \rangle$. That is, every time g hits A, a new bit is encoded according to which element of A is hit.

Question

Given $A \subseteq \omega$ and a transitive model M of ZF, can we create a generic for $\mathbb{H}$ over M without encoding unwanted bits in the process?

By genericity, we **cannot** if $A \in M$. But we **can** if A is infinite but has no infinite subset in M (the Main Lemma)!

Main Theorem: Part 4

Main Lemma

Let M be a transitive model of ZF. Let $A \subseteq \omega$ be infinite but have no infinite subset in M.

Let $\mathbb{P} = \mathbb{H}^M$. Let $\mathcal{D} \in \mathcal{P}^M(\mathbb{P})$ be open dense (in M). Let $T \in \mathbb{P}$.

Then there exists some $T' \leq_A T$ in $\mathcal{D}$.

So if $\mathcal{P}^M(\mathbb{P})$ is countable (in V), then constructing a G which is $\mathbb{P}$-generic over M can be accomplished by constructing a decreasing $\leq_A$-sequence. to hit all ω many dense sets.

Assuming the Main Lemma is true, we can prove the Main Theorem.

Main Theorem: Part 5

So now we can alternate between hitting dense sets by making $\leq_A$-extensions, and encoding whatever bits we want by making non-$\leq_A$-extensions.

Generic Coding with Help Theorem (Main Theorem)

Let M be a transitive model of ZF such that $\mathcal{P}^M(\mathbb{H}^M)$ is countable. Let $\bar{a}, x \in {}^\omega 2$ be such that $\bar{a} \notin M$.

Then there is a G that is $\mathbb{H}^M$-generic over M such that $x \in L[\bar{a}, G]$.

Proof: let $A \subseteq \omega$ be Turing equivalent to $\bar{a}$ and also computable from every infinite subset of itself. Let $\langle \mathcal{D}_i : i < \omega \rangle$ be an enumeration of the open dense subsets of $\mathbb{H}^M$ in M. Let $T'_{-1} = 1 \in \mathbb{H}^M$. Now let $i \geq 0$.

Let $T_i \leq_A T'_{i-1}$ be such that $T_i \in \mathcal{D}_i$. Let $T'_i \leq T_i$ be a non-$\leq_A$-extension of T_i extending the stem of T_i by one to encode the i-th bit of x. etc.

Main Lemma: Part 1

But how do we prove the Main Lemma? That is, how do we $\leq_A$-extend a condition to hit a dense subset $\mathcal{D}$ of $\mathbb{H}^M$ in our countable transitive model M?

Taking one step:

Sticking Out Observation

Let M be a transitive model of ZF. Let $A \subseteq \omega$ be infinite but there are no infinite subsets of A in M. Then if $B \subseteq \omega$ is infinite and in M, then $B - A$ is infinite.

Proof: Assume towards a contradiction that $B - A$ is finite. Then $B - A \in M$. Since both B and $B - A$ are in M, we have $B \cap A \in M$ as well. At the same time, since B is infinite and $B - A$ is finite, $B \cap A$ must be infinite. So now $B \cap A$ is an infinite subset of A which is in A, which is a contradiction.

Main Lemma: Part 2

To prove the Main Lemma, we need a "rank analysis" on the stems of the conditions in our dense set.

Definition

Given $S \subseteq {}^{<\omega}\omega$ and $t \in {}^{<\omega}\omega$,

- t is 0-S-reachable iff $t \in S$;
- t is α-S-reachable for some $\alpha > 0$ iff

$$\{z \in \omega : t^\frown z \text{ is } \beta\text{-}S\text{-reachable for some } \beta < \alpha\}$$

  is infinite.
- t is S-reachable iff t is α-S-reachable for some α.

Fact: fix $T \in \mathbb{H}$ and $\mathcal{D} \subseteq \mathbb{H}$. Let $t = \mathrm{Stem}(T)$ and $S \subseteq {}^{<\omega}\omega$ be the set of stems of elements of $\mathcal{D}$. Then $\mathcal{D}$ is dense below T iff t is S-reachable.

Main Lemma: Part 3

Proof of Main Lemma: fix M and $\mathcal{D} \subseteq \mathbb{H}^M$ which is dense. Fix $T \in \mathbb{H}^M$. Let $A \subseteq \omega$ be infinite with no infinite subsets in M. We will find $T' \leq_A T$ in $\mathcal{D}$. Let $t = \text{Stem}(T)$ and S be the set of stems of elements of $\mathcal{D}$. It suffices to find some $s \sqsupseteq_A t$ in $T \cap S$.

Let α be such that t is α-S-reachable. If $\alpha = 0$, then set $s := t$ and we are done. If not, the set $B = \{z \in \omega : t^\frown z \text{ is } \beta\text{-}S\text{-reachable for some } \beta < \alpha\}$ is infinite (and in M). So by the Sticking Out Observation, $B - A$ is infinite. So, fix some $z_0 \in (B - A)$ such that $t^\frown z_0 \in T$.

Now $t^\frown z_0$ is β-S-reachable for some $\beta < \alpha$. If $\beta = 0$, then set $s := t^\frown z_0$ are we are done. If not, then we can find some $z_1 \notin A$ such that $t^\frown z_0^\frown z_1 \in T$ and $t^\frown z_0^\frown z_1$ is δ-S-reachable for some $\delta < \beta$.

etc. This must eventually terminate.

The complement of a Turing Cone

In the Generic Coding with Help Theorem, we actually have $x \leq_T \bar{a} \oplus g$ where $g = \bigcup \bigcap G$.

Corollary

Assume $0^\#$ exists. Let S be the set of reals that are generic over L. Then although S is disjoint from the Turing cone above $0^\#$, we have that for any $\bar{a} \in {}^\omega 2 - L$, the set

$$\{\bar{a} \oplus s : s \in S\}$$

is cofinal in the Turing degrees.

Nontrivial Nodes of Compatibility?

Definition

Let $\alpha < \omega_1$. The set $\mathcal{H}_\alpha$ is the collection of all (countable) transitive models of ZFC of height α.

We say that $M_1, M_2 \in \mathcal{H}_\alpha$ are **compatible** iff there is some $N \in \mathcal{H}_\alpha$ such that $M_1 \cup M_2 \subseteq N$.

Conjecture (Sy Friedman)

If $M \in \mathcal{H}_\alpha$ is compatible with every model in $\mathcal{H}_\alpha$, then $M = L_\alpha$.

A counterexample would mean there exists some very “gentle” information not in L_α.

Friedman's Conjecture is True

Theorem

Let $M_1, J \in \mathcal{H}_\alpha$ be such that $M_1 \not\subseteq J$. Then there is a forcing extension M_2 of J that is not compatible with M_1.

Proof: let $a' \in M_1 - J$ be a set of ordinals. Force over J to get $J[G_1]$ to make $\sup a'$ countable. Now a' is encoded by a real $\bar{a}$ (which by mutual genericity can be assumed to not be in $J[G_1]$). Then force over $J[G_1]$ to get $M_2 := J[G_1][G_2]$, using the Generic Coding with Help Theorem, so that G_2 together with $\bar{a}$ computes a real not in any model of ZFC of height α.

History: I asked Friedman if the Generic Coding with Help Theorem was already known, and he said no but he was looking for such a theorem to make the proof above work.

Larger Sets are Generically Helpful

The Generic Coding with Help Theorem uses a real number as "help". Larger objects can be used as help when we force over the universe to make them countable.

Corollary

Let M be a transitive model of ZF. Let λ be a cardinal such that $\lambda \in M$. Let $\mathbb{P} = (\mathrm{Col}(\omega, \lambda) * \mathbb{H})^M$. Let $\tilde{V}$ be an outer model of V in which $\mathcal{P}^M(\mathbb{P})$ is countable.

Let $X \in \mathcal{P}^{\tilde{V}}(\lambda)$. Let $A \in \mathcal{P}^{\tilde{V}}(\lambda) - M$. Then there is a G in $\tilde{V}$ such that

1) G is $\mathbb{P}$-generic over M,
2) $X \in L(A, G)$.

Are Larger Sets Helpful in V itself?

The previous slide shows that larger sets are helpful in sufficiently large forcing extensions of the universe. What about in V itself?

Question

Assume CH and a proper class of Woodin cardinals. Is there some $\bar{a} \subseteq \mathbb{R}$ and some forcing $\mathbb{P} \in L(\mathbb{R})$ that is countably closed such that given any $X \subseteq \omega_1$, there is a G that is $\mathbb{P}$-generic over $L(\mathbb{R})$ such that

$$X \in L(\bar{a}, \mathbb{R})[G]?$$

If true, we then ask if it is true for *any* $\bar{a} \subseteq \mathbb{R}$ not in $L(\mathbb{R})$.

Note: Woodin has conjectured that if CH holds and there is a proper class of Woodins and there is a mouse with a measurable Woodin cardinal, then for any $X \subseteq \omega_1$, there is some model M of AD containing all the reals such that X is $\mathrm{Col}(\omega_1, \mathbb{R})$-generic over M.

What about HOD?

Every real is generic over HOD.

Every real is generic over HOD with help using $\mathbb{H}^{\mathrm{HOD}}$.

$\mathbb{H}$ has size 2^ω (and is c.c.c).

Question

Is every real generic over HOD by a poset of size $\leq (2^\omega)^{HOD}$?

Sy Friedman believes he has a proof that the anser is no in some models of ZFC.

The Original Application of the Main Lemma

The original application for the Main Lemma was to prove results like the following:

Theorem (ZF)

Assume there is no injection of ω_1 into $\mathbb{R}$. Fix $a \in \mathbb{R}$. There is a Baire class one function $f_a : {}^\omega\omega \to {}^\omega\omega$ with the following property:

whenever $g : {}^\omega\omega \to {}^\omega\omega$ is ∞-Borel and $f_a \cap g = \emptyset$, then

$$a \in L[C]$$

where C is any ∞-Borel code for g.

Domination of Functions from $\mathbb{R}$ to $\mathbb{R}$

The previous slide gives information about the disjointness relation of functions from $\mathbb{R}$ to $\mathbb{R}$. Given $f, g : \mathbb{R} \to \mathbb{R}$, we say that g everywhere dominates f iff $(\forall x \in \mathbb{R})\, f(x) \leq g(x)$. Here is an aside about the everywhere domination relation:

Theorem

Fix $a \in \mathbb{R}$. Let $f^a : \mathbb{R} \to \mathbb{R}$ be the function

$$f^a(x) = \begin{cases} \frac{1}{(x-a)^2} & \text{if } x \neq a \\ 0 & \text{if } x = a. \end{cases}$$

Suppose $g : \mathbb{R} \to \mathbb{R}$ everywhere dominates f^a.

Then $a \in L[g]$.

Thank You!

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