Disjoint Infinity-Borel Functions Overview

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The statement $\Psi(g)$

Claim

For each $a \in {}^\omega\omega$, there is a Borel function $f_a : {}^\omega\omega \to {}^\omega\omega$ that "encodes a". The following are satisfied:

- $(a, x) \mapsto f_a(x)$ is Borel.
- Large cardinals imply that if $g : {}^\omega\omega \to {}^\omega\omega$ is "nice", then $g \cap f_a = \emptyset$ for only countably many a.

Let $\Psi(g)$ be the statement that $g : {}^\omega\omega \to {}^\omega\omega$ is disjoint from only countably many f_a's.

ZFC implies $(\exists g)\, \neg\Psi(g)$

$\mathrm{ZFC} + \neg\mathrm{CH}$ easily implies $(\exists g)\, \neg\Psi(g)$.

$\mathrm{ZFC} + \mathrm{CH}$ implies there is a set of reals of size 2^ω that cannot be surjected onto ${}^\omega\omega$ by any Borel function. This implies $(\exists g)\, \neg\Psi(g)$.

Δ^1_1 Functions

Theorem

Let $g : {}^\omega\omega \to {}^\omega\omega$ be $\Delta^1_1(c)$ for some $c \in {}^\omega\omega$. Then

$$g \cap f_a = \emptyset \Rightarrow a \in \Delta^1_1(c).$$

Corollary

Let $g : {}^\omega\omega \to {}^\omega\omega$ be $\boldsymbol{\Delta}^1_1$. Then $\Psi(g)$.

The proof of the theorem uses forcing (forcing over an arbitrary ω-model M that contains c but not a to produce $M[x]$ such that $M[x] \models g(x) = f_a(x)$. $M[x]$ understands g because $c \in M$ and g is Borel).

There is a "forcing free" proof with the weaker conclusion that $a \in \Sigma^2_1(c)$. No forcing free proofs are known for the other theorems in this talk.

Δ^1_2 Functions

Theorem

Let g be $\Delta^1_2(c)$ for some $c \in {}^\omega\omega$. Then

$$g \cap f_a = \emptyset \Rightarrow a \in L[c].$$

Corollary

Let g be $\Delta^1_2(c)$ and assume ${}^\omega\omega \cap L[c]$ is countable. Then $\Psi(g)$.

Theorem

The following are equivalent:

- $(\forall g \in \mathbf{\Delta}^1_2)\, \Psi(g)$.
- $(\forall r \in {}^\omega\omega)\, \omega_1$ is inaccessible in $L[r]$.

$(\forall g)\, \Psi(g)$ holds in the Solovay model.

Δ^1_n Functions

Theorem

Assume Projective Determinacy. Let g be $\Delta^1_n(c)$. Then

$$g \cap f_a = \emptyset \Rightarrow a \text{ is } \Delta^1_n \text{ in } c \text{ and a countable ordinal.}$$

Corollary

Assume Projective Determinacy. Then $\Psi(g)$ holds for every projective $g : {}^\omega\omega \to {}^\omega\omega$.

On the next slide, we will see that AD^+ implies $(\forall g)\, \Psi(g)$. Does AD alone imply $(\forall g)\, \Psi(g)$?

Functions in models of AD^+

Theorem (ZF)

Assume there is no injection of ω_1 into ${}^\omega\omega$. Let g be ∞-Borel with code $C \subseteq \mathrm{Ord}$. Then

$$g \cap f_a = \emptyset \Rightarrow a \in L[C].$$

Corollary

AD^+ implies $(\forall g)\, \Psi(g)$.

Corollary

Assume there is a proper class of Woodin cardinals. Let g be universally Baire. Then $\Psi(g)$.

Functions in Forcing Extensions of $L(\mathbb{R})$

PSP is the perfect set property.

Theorem

Let $\mathbb{Q} \in L(\mathbb{R})$ be a forcing such that

- There is a surjection of $\mathbb{R}$ onto $\mathbb{Q}$ in $L(\mathbb{R})$,
- $(\mathbb{Q}$ adds no reals$)^{L(\mathbb{R})}$, and
- $(1 \Vdash_{\mathbb{Q}} PSP)^{L(\mathbb{R})}$.

Then $(1 \Vdash_{\mathbb{Q}} (\forall g)\, \Psi(g))^{L(\mathbb{R})}$.

Corollary

Assume there is a proper class of Woodin cardinals. Let $\mathcal{U}$ be a selective ultrafilter on ω. Then $L(\mathbb{R})[\mathcal{U}] \models (\forall g)\, \Psi(g)$.

Acknowledgments

Paul Larson pointed out the argument that $\mathrm{AC} + \mathrm{add}(\mathcal{M}) = 2^\omega$ implies there is a size 2^ω set of reals that cannot be surjeccted onto $\mathbb{R}$ by a Borel function. He also explained why $L(\mathbb{R})[\mathcal{U}]$ satisfies the perfect set property, which is used in the proof that $L(\mathbb{R})[\mathcal{U}] \models \Psi$. Trevor Wilson explained the large cardinal steps in the proof that Projective Determinacy implies that every projective $g : {}^\omega\omega \to {}^\omega\omega$ is disjoint from at most countably many f_a's.

Thank You!

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