Disjoint Borel Functions

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Cardinal Characteristics

Definition

A challenge-response relation (**c.r.-relation**) is a triple $\langle R_-, R_+, R \rangle$ such that $R \subseteq R_- \times R_+$. The set R_- is the set of **challenges**, and R_+ is the set of **responses**. When cRr, we say that r **meets** c.

Definition

A backwards generalized Galois-Tukey connection (**morphism**) from $\mathcal{A} = \langle A_-, A_+, A \rangle$ to $\mathcal{B} = \langle B_-, B_+, B \rangle$ is a pair $\langle \phi_-, \phi_+ \rangle$ of functions $\phi_- : B_- \to A_-$ and $\phi_+ : A_+ \to B_+$ such that

$$(\forall c \in B_-)(\forall r \in A_+)\, \phi_-(c)\, A\, r \Rightarrow c\, B\, \phi_+(r).$$

When there is a morphism from $\mathcal{A}$ to $\mathcal{B}$, let us say that $\mathcal{A}$ is **above** $\mathcal{B}$ and $\mathcal{B}$ is **below** $\mathcal{A}$

Cardinal Characteristics

Picture showing that $\mathcal{A}$ is above $\mathcal{B}$:

$$\begin{array}{ccccc} A_- & \qquad & A & \qquad & A_+ \\ \phi_- \Big\uparrow & & \Big\Downarrow & & \Big\downarrow \phi_+ \\ B_- & & B & & B_+ . \end{array}$$

Definition

The **norm** of a c.r.-relation $\mathcal{R} = \langle R_-, R_+, R \rangle$ is

$$||\mathcal{R}|| := \min\{|S| : S \subseteq R_+ \text{ and } (\forall c \in R_-)(\exists r \in S)\, c\, R\, r\}.$$

If $\mathcal{A}$ is above $\mathcal{B}$, then $||\mathcal{A}|| \geq ||\mathcal{B}||$.

Main Question

Let $\mathcal{N} = {}^{\omega}\omega$ be Baire space. $\boldsymbol{\Delta}^1_1 \cap {}^{\mathcal{N}}\mathcal{N}$ is the set of Borel functions from $\mathcal{N}$ to $\mathcal{N}$. The results in this talk apply also to $\boldsymbol{\Delta}^1_1 \cap {}^{X}Y$ where X and Y are any Polish spaces with X uncountable.

Throughout this paper, let D be the relation of functions being disjoint. That is, given $f, g : \mathcal{N} \to \mathcal{N}$,

$$f \, D \, g \Leftrightarrow f \cap g = \emptyset \Leftrightarrow (\forall x \in \mathcal{N}) \, f(x) \neq g(x).$$

Main Question

Let $\mathcal{R} = \langle \boldsymbol{\Delta}^1_1 \cap {}^{\mathcal{N}}\mathcal{N}, \boldsymbol{\Delta}^1_1 \cap {}^{\mathcal{N}}\mathcal{N}, D \rangle$.

- What is $||\mathcal{R}||$?
- What c.r.-relations are below $\mathcal{R}$?
- What if we instead look at $\boldsymbol{\Delta}^1_n \cap {}^{\mathcal{N}}\mathcal{N}$ for $n \geq 2$ (and beyond)?

Basic Plan

Let $\mathcal{R} = \langle \mathbf{\Delta}^1_1 \cap {}^{\mathcal{N}}\mathcal{N}, \mathbf{\Delta}^1_1 \cap {}^{\mathcal{N}}\mathcal{N}, D \rangle$.

Let $\prec$ be an ordering on $\mathcal{N}$ such that

$$(\forall r \in \mathcal{N})\, \{a \in \mathcal{N} : a \prec r\} \text{ is countable.}$$

If we can show that there is a morphism from $\mathcal{R}$ to $\langle \mathcal{N}, \mathcal{N}, \prec \rangle$, we will have that $||\mathcal{R}|| = 2^\omega$. What ordering $\prec$ will work?

- $a \prec r \Leftrightarrow a \leq_T r$?
- $a \prec r \Leftrightarrow a \in \Delta^1_1(r)$?
- $a \prec r \Leftrightarrow a \in \Delta^1_2(r)$?
- $a \prec r \Leftrightarrow a \in L(r)$?
- $a \prec r \Leftrightarrow a \in \Delta^1_3(r)$?
- $a \prec r \Leftrightarrow a \in \mathcal{M}_1(r)$?
- ...
- $a \prec r \Leftrightarrow a \in HOD(r)$?

The Morphism Functions

Before we figure out which ordering $\prec$ will work, let us say what the functions in the morphism $\langle \phi_-, \phi_+ \rangle$ from $\mathcal{R}$ to $\langle \mathcal{N}, \mathcal{N}, \prec \rangle$ will be.

$$\begin{array}{ccccc} \mathbf{\Delta}^1_1 \cap {}^{\mathcal{N}}\mathcal{N} & & D & & \mathbf{\Delta}^1_1 \cap {}^{\mathcal{N}}\mathcal{N} \\ \phi_- \big\uparrow & & \Big\Downarrow & & \big\downarrow \phi_+ \\ \mathcal{N} & & \prec & & \mathcal{N}. \end{array}$$

ϕ_+ will simply map a function g to any (Borel) code for g.

ϕ_- will map a real a to the Baire class one (pointwise limit of continuous, and therefore Borel) function $f_a \in {}^{\mathcal{N}}\mathcal{N}$ which we will define on the next slide.

This same pair $\langle \phi_-, \phi_+ \rangle$ will also be a morphism from $\langle \mathbf{\Delta}^1_n \cap {}^{\mathcal{N}}\mathcal{N}, \mathbf{\Delta}^1_n \cap {}^{\mathcal{N}}\mathcal{N}, D \rangle$ to $\langle \mathcal{N}, \mathcal{N}, \prec \rangle$ for an appropriate $\prec$ corresponding to n.

The Encoding Function f_a

Fix $a \in \mathcal{N}$. Pick some $A \subseteq \omega$ such that $A =_T a$, A is infinite, and $A \leq_T B$ whenever B is an infinite subset of A. Such a set A is easy to construct. We actually only need A to be Δ^1_1 in every infinite subset of itself.

Let $h : A \to \omega$ be a function such that $(\forall n \in \omega)\, h^{-1}(n)$ is infinite.

We will now define $f_a : \mathcal{N} \to \mathcal{N}$. Fix $x = \langle x_0, x_1, ...\rangle \in \mathcal{N}$. Let $i_0 < i_1 < ...$ be the sequence of indices listing which numbers x_i are in A. That is, each $x_{i_k} \in A$, but no other x_i is in A. Define

$$f_a(x) := \langle h(x_{i_0}), h(x_{i_1}), ...\rangle$$

If there are only finitely many x_i in A, define $f_a(x)$ to be anything.

The function f_a is Baire class one (and therefore $\mathbf{\Delta}^1_1$).
Fact: no continuous function will work as an encoding function (which ultimately follows from the fact that the set of well-founded subtrees of ${}^{<\omega}\omega$ ordered by inclusion has cofinality $\mathfrak{d}$).

Variant of Hechler Forcing

Given an appropriate ordering $\prec$ on $\mathcal{N}$, to show that $\langle \phi_-, \phi_+ \rangle$ is indeed a morphism from $\mathcal{R}$ to $\langle \mathcal{N}, \mathcal{N}, \prec \rangle$ we will perform a forcing argument!!!

Definition

Fix $h : {}^{<\omega}\omega \to \omega$, $A \subseteq \omega$, and $t_1, t_2 \in {}^{<\omega}\omega$.

- $t_2 \sqsupseteq_h t_1$ iff $t_2 \sqsupseteq t_1$ and $(\forall n \in \mathrm{Dom}(t_2) - \mathrm{Dom}(t_1))\, t_2(n) \geq h(t_2 \restriction n)$.
- $t_2 \sqsupseteq_h^A t_1$ iff $t_2 \sqsupseteq_h t_1$ and $(\forall n \in \mathrm{Dom}(t_2) - \mathrm{Dom}(t_1))\, t_2(n) \notin A$.

That is, $t_2 \sqsupseteq_h t_1$ means that t_2 is an extension of t_1 "to the right" of h, and $t_2 \sqsupseteq_h^A t_1$ means that additionally t_2 does not "hit" A any more than t_1 already does.

Given $h_1, h_2 : {}^{<\omega}\omega \to \omega$, let us write $h_2 \geq h_1$ iff $(\forall t \in {}^{<\omega}\omega)\, h_2(t) \geq h_1(t)$.

Definition

$\mathbb{H}$ is the poset of all pairs (t, h) such that $t \in {}^{<\omega}\omega$ and $h : {}^{<\omega}\omega \to \omega$, where $(t_2, h_2) \leq (t_1, h_1)$ iff $t_2 \sqsupseteq_{h_1} t_1$ and $h_2 \geq h_1$. Given $A \subseteq \omega$, we write $(t_2, h_2) \leq^A (t_1, h_1)$ iff $t_2 \sqsupseteq_{h_1}^A t_1$ and $h_2 \geq h_1$.

The Main Lemma

Main Lemma

Let M be an ω-model of ZF and $U \in \mathcal{P}^M(\mathbb{H}^M)$ be a set dense in $\mathbb{H}^M$. Let $A \subseteq \omega$ be infinite and Δ^1_1 in every infinite subset of itself but $A \notin M$. Then

$$(\forall p \in \mathbb{H}^M)(\exists p' \leq^A p)\, p' \in U.$$

Note: $(\forall x, y \in \mathcal{N})\, x \in \Delta^1_1(y)$ iff every ω-model M which contains y also contains x.

Thus, letting M, U, and A satisfy the hypothesis of the lemma, then defining

$$S := \{t \in {}^{<\omega}\omega : (\exists h \in M)\,(t, h) \in U\},$$

we have $S \in M$ and therefore $A \notin \Delta^1_1(S)$.

Proof of Main Lemma

We can prove the main lemma by a rank analysis.

Definition

Given $t \in {}^{<\omega}\omega$ and $S \subseteq {}^{<\omega}\omega$,

- t is 0-S-**reachable** iff $t \in S$;
- for $\alpha > 0$, t is α-S-**reachable** iff t is β-S-reachable for some $\beta < \alpha$ or $\{n \in \omega : (\exists \beta < \alpha)\, t^\frown n \text{ is } \beta\text{-}S\text{-reachable}\}$ is infinite.
- t is S-**reachable** iff t is α-S-reachable for some α.

A computation shows the following:

- t is S-reachable iff t is α-S-reachable for some $\alpha < \omega_1^{CK}(S)$.
- Given $\alpha < \omega_1^{CK}(S)$, the set of all t that are β-S-reachable for some $\beta < \alpha$ is $\Delta^1_1(S)$.

Proof of Main Lemma

The main lemma follows at once from the following:

Lemma (Reachability Dichotomy)

Fix $t \in {}^{<\omega}\omega$, $S \subseteq {}^{<\omega}\omega$, and $A \subseteq \omega$ which is infinite and Δ^1_1 in every infinite subset of itself. Assume $A \notin \Delta^1_1(S)$.

- If t is not S-reachable, then
$$(\exists h \in \Delta^1_1(S))(\forall t' \sqsupseteq_h t)\, t' \notin S.$$
- If t is S-reachable, then
$$(\forall h)(\exists t' \sqsupseteq^A_h t)\, t' \in S.$$

The first case follows easily from the fact that if t is **not** S-reachable, then only finitely many $t^\frown n$ **are** S-reachable. For each t that is not S-reachable, define $h(t)$ to be the smallest n such that $(\forall m \geq n)\, t^\frown m$ is not S-reachable. A computation shows that $h \in \Delta^1_1(S)$.

Proof of Main Lemma

We will sketch a proof of the second case of the reachability dichotomy. Fix t,S, and A as in that lemma. Assume that t is S-reachable and fix $h : {}^{<\omega}\omega \to \omega$. We must find some $t' \sqsupseteq_h^A t$ such that $t' \in S$.

Assume that t is not 0-S-reachable, otherwise we are already done by setting $t' = t$. Thus, fix the smallest $\alpha > 0$ such that t is α-S-reachable.

By induction, it suffices to find some $n \in \omega$ such that $n \notin A$, $n \geq h(t)$, and $t^\frown n$ is β-S-reachable for some $\beta < \alpha$. Let

$$B := \{n \in \omega : (\exists \beta < \alpha)\, t^\frown n \text{ is } \beta\text{-}S\text{-reachable}\}.$$

B is infinite and $B \in \Delta^1_1(S)$. If $B - A$ is infinite, we can get the desired n. Now, $B - A$ must be infinite because otherwise $B \cap A =_T B$ and $B \cap A$ is infinite, so

$$A \leq_{\Delta^1_1} B \cap A =_T B \leq_{\Delta^1_1} S,$$

which implies $A \leq_{\Delta^1_1} S$, a contradiction. □

Main Theorem

Main Theorem

Let Γ be the pointclass of all sets defined by formulas in a certain class (so it makes sense to talk about a Γ-formula).

Let $\prec$ be an ordering on $\mathcal{N}$ such that whenever $r, a \in \mathcal{N}$ are such that $a \not\prec r$, then there exists an ω-model M of ZF such that

- $r \in M$;
- $a \notin M$;
- $\mathcal{P}^M(\mathbb{H}^M)$ is countable (in V);
- for every forcing extension N (in V) of M by $\mathbb{H}^M$, N can compute the truth (in V) of Γ-formulas with the real param r.

Then for any $a \in \mathcal{N}$ and $g \in \Gamma \cap {}^{\mathcal{N}}\mathcal{N}$,

$$f_a \cap g = \emptyset \quad \Rightarrow \quad a \prec \text{(any code for } g\text{)}.$$

Proof of Main Theorem (from Main Lemma)

Fix a, g, and an arbitrary code r for g. In any model N that contains r, let $\tilde{g}$ refer to $g \cap N$. Since the forcing extension N we will construct will compute the truth of Γ-formulas with the real param r, we will have $\tilde{g} \in N$. Suppose $a \not\leq r$. Fix an ω-model M as in the hypothesis of the theorem. Let $A \subseteq \omega$ be the set from the definition of f_a that is Δ^1_1 in every infinite subset of itself and $a =_T A$. Note that $A \notin M$.

We will construct an $x \in \mathcal{N}$ satisfying $f_a(x) = g(x)$. Let $\langle U_n : n < \omega \rangle$ be an enumeration (in V) of the dense subsets of $\mathbb{H}^M$ in M. Let $\dot{x}$ be the canonical name for the generic real added by $\mathbb{H}^M$. We will construct a decreasing sequence of conditions of $\mathbb{H}^M$ which will hit each U_n. The $x \in \mathcal{N}$ will be the union of the stems in this sequence (and it will be generic over M having the name $\dot{x}$).

Proof of Main Theorem (from Main Lemma)

Starting with $1 \in \mathbb{H}^M$, apply the main lemma to get $p_0 \leq^A 1$ in U_0. Then, apply the main lemma to get $p_0' \leq^A p_0$ and $m_0 \in \omega$ such that $(p_0' \Vdash \tilde{g}(\dot{x})(0) = \check{m}_0)^M$. Next, extend the stem of p_0' by one to get $p_0'' \leq p_0'$ to ensure that $f_a(x)(0) = m_0$.

Next, get $p_1'' \leq p_1' \leq^A p_1 \leq^A p_0''$ such that $p_1 \in U_1$, $(p_1' \Vdash \tilde{g}(\dot{x})(1) = \check{m}_1)^M$ for some $m_1 \in \omega$, and p_1'' extends the stem of p_1' by one to ensure that $f_a(x)(1) = m_1$. And so on...

The x we have constructed is generic for $\mathbb{H}^M$ over M. Let $N = M[x]$. For each $n \in \omega$ we have $(\tilde{g}(x)(n) = m_n)^N$. Since $\tilde{g} = N \cap g$, for each $n \in \omega$ we have

$$g(x)(n) = m_n.$$

On the other hand, for each $n \in \omega$ we have $f_a(x)(n) = m_n$. $\square$

Corollary

Fix $a \in \mathcal{N}$, Γ, $g \in \Gamma \cap {}^{\mathcal{N}}\mathcal{N}$, and a code r for g. Assume $f_a \cap g = \emptyset$.

- $\Gamma = \mathbf{\Delta}^1_1 \Rightarrow a \in \Delta^1_1(r)$;
- $\Gamma = \mathbf{\Delta}^1_2 \Rightarrow a \in L(r)$;
- $\Gamma = \mathbf{\Delta}^1_3 \Rightarrow a \in \mathcal{M}_1(r)$;
- $\Gamma = \mathbf{\Delta}^1_4 \Rightarrow a \in \mathcal{M}_2(r)$;
- ...
- $\Gamma = \mathrm{HOD}^{L(\mathbb{R})} \Rightarrow a \in \mathcal{M}_\omega(r)$;

The first bullet holds because Δ^1_1 formulas are absolute between ω-models and V, and whenever $a \notin \Delta^1_1(r)$, there is some ω-model of ZF which contains r but not a.

The model $\mathcal{M}_1(r)$ can compute the truth of every $\Delta^1_3(r)$ formula in every forcing extension of size below its bottom Woodin cardinal. See [Steel].

Restatement of Corollary

Using facts about what reals are in the relevant models, we have the following:

Corollary

Fix $a \in \mathcal{N}$, Γ, $g \in \Gamma \cap {}^{\mathcal{N}}\mathcal{N}$, and a code r for g. Assume $f_a \cap g = \emptyset$.

- $\Gamma = \boldsymbol{\Delta}^1_1 \Rightarrow a$ is Δ^1_1 in r;
- $\Gamma = \boldsymbol{\Delta}^1_2 \Rightarrow a$ is Δ^1_2 in r and a countable ordinal;
- $\Gamma = \boldsymbol{\Delta}^1_3 \Rightarrow a$ is Δ^1_3 in r and a countable ordinal;
- ...

ZFC proof for projective functions?

Question

Does ZFC prove that for every $a \in \mathcal{N}$ there is some projective $f'_a \in {}^{\mathcal{N}}\mathcal{N}$ and for every projective $g \in {}^{\mathcal{N}}\mathcal{N}$ there is some countable $G(g) \subseteq \mathcal{N}$ such that $(\forall a \in \mathcal{N})(\forall g \in {}^{\mathcal{N}}\mathcal{N}$ projective),

$$f'_a \cap g = \emptyset \quad \Rightarrow \quad a \in G(g)?$$

Perhaps this is the wrong question to ask.

Question

Does ZFC prove the statement in the above question but with the additional requirement that the function $(a, x) \mapsto f'_a(x)$ is projective?

No. It is false in any model in which there is a projective well-ordering of $\mathcal{N}$ and $\omega_2 \leq \mathfrak{b}$.

Arbitrary functions?

Question

Does ZFC + large cardinals imply that for every $a \in \mathcal{N}$ there is some $f'_a \in {}^{\mathcal{N}}\mathcal{N}$ and for **every** $g \in {}^{\mathcal{N}}\mathcal{N}$ there is some countable set $G(g) \subseteq \mathcal{N}$ such that $(\forall a \in \mathcal{N}, g \in {}^{\mathcal{N}}\mathcal{N})$

$$f_a \cap g = \emptyset \quad \Rightarrow \quad a \in G(g)?$$

No. This is false assuming $\mathrm{ZFC} + \neg\mathrm{CH}$: Given $\mathcal{A} \subseteq \mathcal{N}$ of size ω_1, consider $\{f'_a : a \in \mathcal{A}\}$. There must be a $g \in {}^{\mathcal{N}}\mathcal{N}$ disjoint from each f'_a for $a \in \mathcal{A}$. However, it cannot be that $\mathcal{A} \subseteq G(g)$. If $(a, x) \mapsto f'_a(x)$ is Borel, the statement is also false assuming $\mathrm{ZFC} + \mathrm{CH}$ using a diagonalization argument.

Arbitrary functions for encoding subsets of $\mathcal{N}$?

Question

Does ZFC + large cardinals imply that there is some $\lambda < 2^{\mathfrak{c}}$ and for every $A \subseteq \mathcal{N}$ there is some $f'_A \in {}^{\mathcal{N}}\mathcal{N}$ and for every $g \in {}^{\mathcal{N}}\mathcal{N}$ there is some set $G(g) \subseteq \mathcal{P}(\mathcal{N})$ of size λ such that $(\forall A \subseteq \mathcal{N}, g \in {}^{\mathcal{N}}\mathcal{N})$

$$f'_A \cap g = \emptyset \quad \Rightarrow \quad A \in G(g)?$$

No. This cannot hold in any model in which both $\mathfrak{b} = \mathfrak{d} = \mathfrak{c}$ and $\mathrm{cf}\langle {}^{\mathfrak{c}}\mathfrak{c}, \leq \rangle < 2^{\mathfrak{c}}$, and this can be forced by a small poset.

This contrasts with the situation for the everywhere domination relation $\leq$ of functions from $\mathcal{N}$ to ω. Fact: for each $a \in \mathcal{N}$ there is a Baire class one $f'_a : \mathcal{N} \to \omega$ such that whenever $g : \mathcal{N} \to \omega$ is any function satisfying $f'_a \leq g$, then a is Δ^1_1 definable using g as a predicate. Also, for each $A \subseteq \mathcal{N}$ there is an $f'_A : \mathcal{N} \to \omega$ such that whenever $g : \mathcal{N} \to \omega$ is any function satisfying $f'_A \leq g$, then A is $\mathbf{\Delta}^1_1$ definable using g as a predicate.

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Thank You!