Generalized Domination

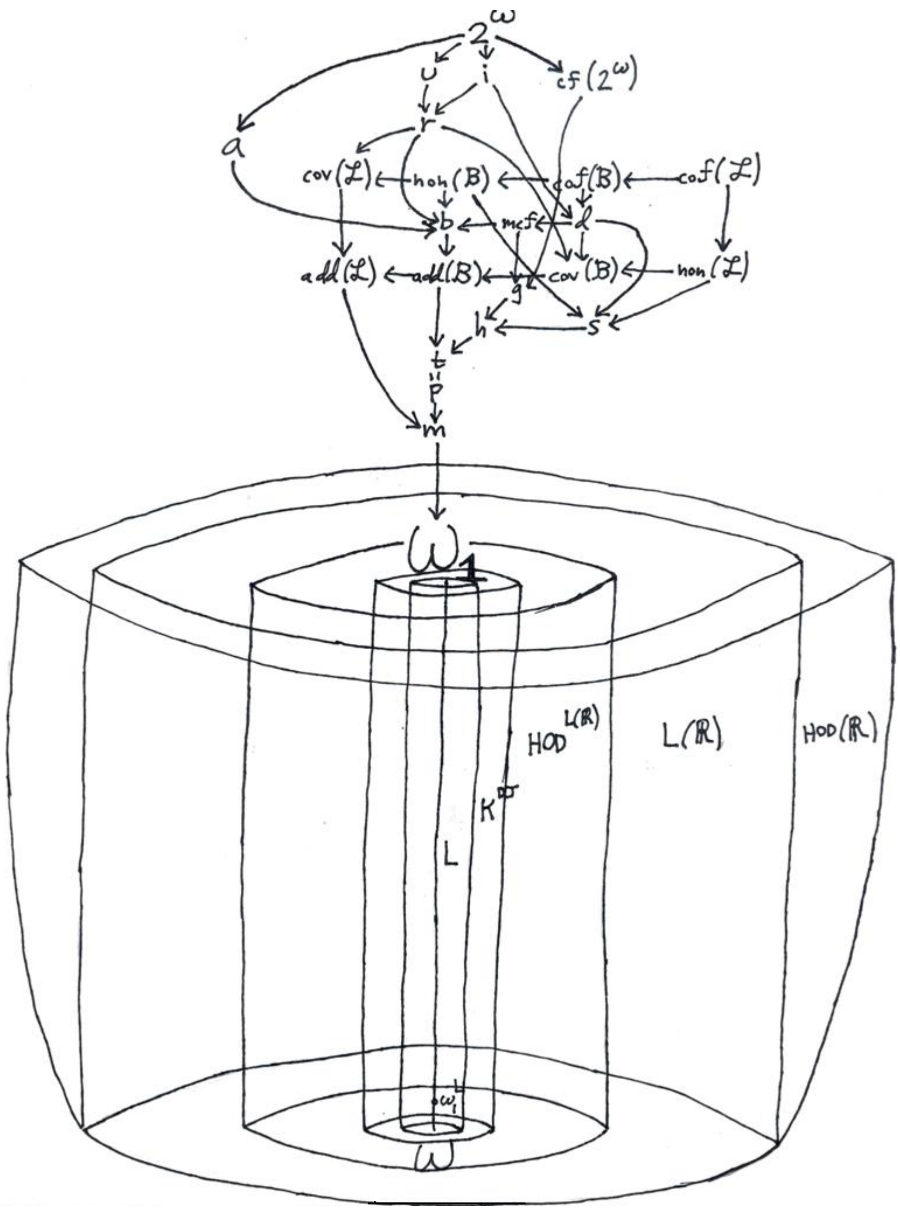
by

Daniel J. Hathaway

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2015

Doctoral Committee:

Professor Andreas R. Blass, Chair
Professor Sergey Fomin
Professor Emeritus Peter G. Hinman
Associate Professor Yaoyun Shi
Professor Karen E. Smith



2^ω
u
i
cf(2^ω)
r
a
cov($\mathcal{L}$)
non($\mathcal{B}$)
cof($\mathcal{B}$)
cof($\mathcal{L}$)
b
mcf
d
add($\mathcal{L}$)
add($\mathcal{B}$)
g
cov($\mathcal{B}$)
non($\mathcal{L}$)
h
s
t
p
m
ω_1
$\mathrm{HOD}^{L(\mathbb{R})}$
$L(\mathbb{R})$
$\mathrm{HOD}(\mathbb{R})$
K^{DJ}
L
ω_1^L
ω

To the trees of Ann Arbor (which grow upwards).

ACKNOWLEDGEMENTS

I would like to thank my friends in Ann Arbor: Will Drobny, David Prigge, Chris Fraser, Alex Mueller, and Becky Hoai. I would also like to thank Karen Smith for her encouragement and support. Peter Hinman was more than kind to read through this thesis and check for errors. His book on mathematical logic was instrumental in my choice to study logic. Lastly, I would like to thank Andreas Blass for his inspiration, undying patience, and wonderful explanations.

TABLE OF CONTENTS

LIST OF APPENDICES

Appendix

CHAPTER I

Introduction

Within this chapter, we will summarize the results of this thesis. We will also provide the main definitions needed to understand the results. We will be using standard notations from set theory, although we will remind the reader of some basic definitions.

1.1 The Overall Program

A fundamental problem in infinitary combinatorics is to compute the *cofinality* of partially ordered sets (posets):

Definition I.1. Given a poset $\mathbb{P} = \langle X, \leq \rangle$, a set $A \subseteq X$ is *cofinal* in $\mathbb{P}$ if

$$(\forall x \in X)(\exists a \in A)\, x \leq a.$$

The *cofinality* of $\mathbb{P}$ is defined as

$$\operatorname{cf} \mathbb{P} := \min\{|A| : A \subseteq X \text{ is cofinal in } \mathbb{P}\}.$$

We abuse terminology by calling $\langle X, \leq \rangle$ a poset whenever $\leq$ is a binary relation that is reflexive and transitive. That is, we do not insist on antisymmetry, so what we call posets should technically be called pre-ordered sets. A cofinal subset of a poset is also sometimes called a dominating family. Given two subsets A, B of a poset, we say that B dominates A if $(\forall a \in A)(\exists b \in B)\, a \leq b$.

As an example of why we would want to compute the cofinality of a poset, it is true that for any infinite cardinals λ and κ,

$$\lambda^\kappa = 2^\kappa \cdot \mathrm{cf}\langle [\lambda]^\kappa, \subseteq \rangle$$

where $[\lambda]^\kappa$ is the set of all size κ subsets of λ. The laws of cardinal exponentiation are not fully understood, and computing the cofinality of $\langle [\lambda]^\kappa, \subseteq \rangle$ turns out to be a useful way to compute λ^κ. The study of the cofinalities of partially ordered sets is fundamental to Shelah's *PCF Theory* ([41], [4], [23]), which is a powerful tool for proving results about cardinal exponentiation.

Let ω be the set of natural numbers (otherwise known as the first infinite ordinal, which is also the first infinite cardinal $\aleph_0$). Let ω_1 be the set of countable ordinals (otherwise known as the second infinite cardinal, the first uncountable cardinal $\aleph_1$). Let 2^ω be the cardinality of $\mathbb{R}$ (the first ordinal which can be bijected with $\mathbb{R}$). A ubiquitous partially ordered set is the set of functions from ω to ω ordered by *everywhere domination*:

$$f \leq g :\Leftrightarrow (\forall x \in \omega)\, f(x) \leq g(x).$$

The cofinality of this poset is denoted $\mathfrak{d}$, the *dominating number*. It is consistent with ZFC that $\omega_1 < \mathfrak{d} < 2^\omega$. This number arises naturally in various contexts. For an exposition of this and related cardinals, see [2]. A closely related poset is $\langle {}^\omega\omega, \leq^* \rangle$, where ${}^\omega\omega$ is the set of functions from ω to ω and $\leq^*$ is defined as follows:

$$f \leq^* g :\Leftrightarrow (\forall^\infty n) f(n) \leq g(n).$$

By $(\forall^\infty n)$ we mean "for all but finitely many n", and by $(\exists^\infty n)$ we mean "there exist infinitely many n". It is not hard to see that $\mathrm{cf}\, \langle {}^\omega\omega, \leq^* \rangle = \mathfrak{d}$.

More generally, given infinite cardinals λ and κ, one can consider the poset of functions from λ to κ ordered by *everywhere domination*:

$$f \leq g :\Leftrightarrow (\forall x \in \lambda)\, f(x) \leq g(x).$$

We denote this poset by $\langle {}^{\lambda}\kappa, \leq \rangle$. If we only care about the cofinality of this poset, then without loss of generality κ is a regular cardinal and $\kappa \leq \lambda$. Computing this cofinality turns out to be highly problematic. It is currently unknown whether ZFC proves $\mathrm{cf}\,\langle {}^{\omega_1}\omega, \leq \rangle = 2^{\omega_1}$. One might conjecture $\mathrm{cf}\,\langle {}^{\lambda}\kappa, \leq \rangle = 2^{\lambda}$ whenever $\kappa < \lambda$, but this is false when there exists a real-valued measurable cardinal [43]. However, if $\lambda^{\kappa} = \lambda$, then $\mathrm{cf}\,\langle {}^{\lambda}\kappa, \leq \rangle = 2^{\lambda}$. This follows from the classical result (see the end of Chapter 3 of [5]) that when $\lambda^{\kappa} = \lambda$, there exists a sufficiently *independent* family of 2^{λ} functions from λ to κ.

The first instance of the equality $\lambda^{\kappa} = \lambda$ is when $\lambda = 2^{\omega}$ and $\kappa = \omega$. In this situation, we might as well be studying the poset of functions from $\mathbb{R}$ to ω ordered by everywhere domination. The cofinality of this poset is $2^{2^{\omega}}$, but there is more detailed information we might want to know. For example, if we restrict our attention to those functions which are *Borel*, will the cofinality still be as large as possible? Answering such a question requires us to develop new techniques. These techniques in turn yield results which are interesting in their own right, such as the following: for each $A \subseteq \mathbb{R}$, there is a function $f : \mathbb{R} \to \omega$ such that if $g : \mathbb{R} \to \omega$ everywhere dominates f, then $A \in L(\mathbb{R}, g)$. The class $L(\mathbb{R}, g)$ is the smallest transitive model of ZF containing all the ordinals, $\mathbb{R}$, and g.

This thesis explores the following idea: we may show that the cofinality of a poset $\langle X, \leq \rangle$ is large by showing that information can be "encoded" into elements of X in such a way that information can also be decoded from any larger elements in X. That is, we may show that $\mathrm{cf}\,\langle X, \leq \rangle$ is large by proving an appropriate "infinite coding theorem". We will explain with an example:

Let X be the set of all functions from $\mathbb{R}$ to ω, and let $\leq$ be the everywhere domination ordering. Suppose Alice has a message $A \subseteq \omega$ which she wants to send to Bob. There exists a way that Alice can "encode" A into a Baire class one (and therefore Borel) function $f_A : \mathbb{R} \to \omega$. Alice wants to give Bob the function f_A, but instead an enemy steps in and substitutes a function $g : \mathbb{R} \to \omega$, which everywhere dominates f_A, and gives this to Bob instead. There is no way that Bob can uniquely recover the original message. This is because if A_1 and A_2 are two different messages, and f_{A_1} and f_{A_2} are encoding A_1 and A_2 respectively, then the enemy can create the function g defined by $g(x) := \max\{f_{A_1}(x), f_{A_2}(x)\}$. Given g, Bob has no way of knowing whether A_1 or A_2 was the original message. However, Bob can guess A by making only countably many guesses. Specifically, A will be one of the (countably many) sets which are Δ^1_1 definable using a predicate for g. This is a prototypical example of a result we will prove.

This thesis is organized according to this theme of coding. We will analyze various situations and determine whether or not such coding results exist.

1.2 Generalized Galois-Tukey Connections (Morphisms)

Before we discuss Galois-Tukey connections, let us define another important concept relevant to the study of posets:

Definition I.2. Given a poset $\mathbb{P} = \langle X, \leq \rangle$, a set $A \subseteq X$ is *unbounded* in $\mathbb{P}$ if

$$(\forall x \in X)(\exists a \in A)\, a \not\leq x.$$

The *bounding number* of $\mathbb{P}$ is defined as

$$\mathfrak{b}\,\mathbb{P} := \{|A| : A \subseteq X \text{ is unbounded in } \mathbb{P}\}.$$

A set which is not unbounded is *bounded.* Sometimes the cofinality $\mathrm{cf}\,\mathbb{P}$ of a poset $\mathbb{P}$ is denoted $\mathfrak{d}\,\mathbb{P}$ and is called the dominating number, to accompany the terminology for the bounding number.

The class of all partially ordered sets can itself be (pre)ordered by the *Tukey ordering*:

Definition I.3. The poset $\mathbb{P} = \langle P, \leq_P \rangle$ is *Tukey above* the poset $\mathbb{Q} = \langle Q, \leq_Q \rangle$ if there exists a pair $\langle \phi_-, \phi_+ \rangle$ of functions such that $\phi_- : Q \to P$, $\phi_+ : P \to Q$, and

$$(\forall q \in Q)(\forall p \in P)[\phi_-(q) \leq_P p \Rightarrow q \leq_Q \phi_+(p)].$$

The pair $\langle \phi_-, \phi_+ \rangle$ is called a *Galois-Tukey connection* from $\mathbb{P}$ to $\mathbb{Q}$.

When both $\mathbb{P}$ is Tukey above $\mathbb{Q}$ and $\mathbb{Q}$ is Tukey above $\mathbb{P}$, we say that $\mathbb{P}$ and $\mathbb{Q}$ have the same *Tukey type*, although we will not need this definition. When $\langle P, \leq_P \rangle$ is Tukey above $\langle Q, \leq_Q \rangle$, we may depict this using a diagram as follows:

$$\begin{array}{ccc} P & \leq_P & P \\ \uparrow & \Downarrow & \downarrow \\ Q & \leq_Q & Q. \end{array}$$

Moreover, when this is witnessed by the Galois-Tukey connection $\langle \phi_-, \phi_+ \rangle$, we may depict this by labeling the appropriate arrows in the diagram:

$$\begin{array}{ccc} P & \leq_P & P \\ {\scriptstyle \phi_-}\uparrow & \Downarrow & \downarrow{\scriptstyle \phi_+} \\ Q & \leq_Q & Q. \end{array}$$

It turns out that the following are equivalent for posets $\mathbb{P} = \langle P, \leq_P \rangle$ and $\mathbb{Q} = \langle Q, \leq_Q \rangle$:

1) $\mathbb{P}$ is Tukey above $\mathbb{Q}$;

2) There exists a function $f : P \to Q$ which maps sets cofinal in $\mathbb{P}$ to sets cofinal in $\mathbb{Q}$;

3) There exists a function $g : Q \to P$ which maps sets unbounded in $\mathbb{Q}$ to sets unbounded in $\mathbb{P}$.

If $\langle g, f\rangle$ is a Galois-Tukey connection that witnesses that $\mathbb{P}$ is Tukey above $\mathbb{Q}$, then f witnesses that 2) is true, and g witnesses that 3) is true. If f witnesses that 2) is true, then there exists a g such that $\langle g, f\rangle$ witnesses that $\mathbb{P}$ is Tukey above $\mathbb{Q}$. An analogous statement can be made for 3). Calling a Galois-Tukey connection from $\mathbb{P}$ to $\mathbb{Q}$ a *morphism* from $\mathbb{P}$ to $\mathbb{Q}$, we have that the class of posets forms a category. This is sometimes called the *Tukey category*. In this thesis, when we talk about a morphism from one poset to another, we mean this notion.

As an example, if $\kappa \leq \lambda_1 < \lambda_2$, then there is a morphism from $\langle {}^{\lambda_2}\kappa, \leq\rangle$ to $\langle {}^{\lambda_1}\kappa, \leq\rangle$. However, if $\kappa_1 < \kappa_2 \leq \lambda$, there is no obvious reason why there should be a morphism in either direction between $\langle {}^{\lambda}\kappa_1, \leq\rangle$ and $\langle {}^{\lambda}\kappa_2, \leq\rangle$.

The existence of a morphism from $\mathbb{P}$ to $\mathbb{Q}$ gives us useful information. Most importantly, we have the following:

Observation I.4. *If there is a morphism from* $\mathbb{P} = \langle P, \leq_P\rangle$ *to* $\mathbb{Q} = \langle Q, \leq_Q\rangle$*, then*

1) $\operatorname{cf}\mathbb{Q} \leq \operatorname{cf}\mathbb{P}$*;*

2) $\mathfrak{b}\,\mathbb{P} \leq \mathfrak{b}\,\mathbb{Q}$.

In the next section, we will see a few more consequences of the existence of a morphism. Let us give a classical example of the existence of a morphism. Recall that $\Delta^1_1 \cap \mathcal{P}(\omega)$ is the set of *hyperarithmetical* subsets of ω. As a consequence of [28] and [42], there exists a morphism from $\langle \Delta^1_1 \cap {}^\omega\omega, \leq \rangle$ to $\langle \Delta^1_1 \cap \mathcal{P}(\omega), \leq_T \rangle$:

$$\begin{array}{ccc} \Delta^1_1 \cap {}^\omega\omega & \leq & \Delta^1_1 \cap {}^\omega\omega \\ \uparrow & \Downarrow & \downarrow \\ \Delta^1_1 \cap \mathcal{P}(\omega) & \leq_T & \Delta^1_1 \cap \mathcal{P}(\omega). \end{array}$$

We will describe this morphism in Section 2.8. The relation $\leq_T$ is Turing reducibility, also called relative recursiveness. That is, $a \leq_T b$ iff a is computable by a Turing machine which uses b as an oracle.

This is an example of a connection between the domination relation and computability theory. In this thesis, we find more connections of this sort.

What we have said can be generalized beyond posets to *challenge-response relations*:

Definition I.5. A *challenge-response relation* is a triple $\langle R_-, R_+, R \rangle$ such that $R \subseteq R_- \times R_+$. The set R_- is the set of *challenges*. The set R_+ is the set of *responses*. When cRr, we say that r *meets* c.

There is the appropriate generalization of Galois-Tukey connection:

Definition I.6. Given the challenge-response relations $\mathcal{A} = \langle A_-, A_+, A \rangle$ and $\mathcal{B} = \langle B_-, B_+, B \rangle$, we call $\langle \phi_-, \phi_+ \rangle$ a *generalized Galois-Tukey connection* from $\mathcal{A}$ to $\mathcal{B}$ if $\phi_- : B_- \to A_-$, $\phi_+ : A_+ \to B_+$, and

$$(\forall b \in B_-)(\forall a \in A_+)\, \phi_-(b) A a \Rightarrow b B \phi_+(a).$$

As before, we may depict that $\langle \phi_-, \phi_+ \rangle$ is a generalized Galois-Tukey connection by the following diagram:

$$\begin{array}{ccc} A_- & A & A_+ \\ {\scriptstyle \phi_-}\uparrow & \Downarrow & \downarrow{\scriptstyle \phi_+} \\ B_- & B & B_+. \end{array}$$

Also as before, the class of challenge-response relations forms a category with generalized Galois-Tukey connections as the morphisms. In this thesis, when we talk about a morphism from one challenge-response relation to another, we mean this notion.

The analogue of the cofinality of a poset is the *norm* of a challenge-response relation:

Definition I.7. Given a challenge-response relation $\mathcal{R} = \langle R_-, R_+, R\rangle$, a set $A \subseteq R_+$ is *adequate* for $\mathcal{R}$ if

$$(\forall x \in R_-)(\exists a \in A)\, xRa.$$

The *norm* of $\mathcal{R}$ is defined as

$$||\mathcal{R}|| := \min\{|A| : A \subseteq R_+ \text{ is adequate for } \mathcal{R}\}.$$

Every poset $\langle P, \leq_P\rangle$ can be viewed as a challenge-response relation $\langle P, P, \leq_P\rangle$. We have that $\operatorname{cf}\langle P, \leq_P\rangle = ||\langle P, P, \leq_P\rangle||$. A morphism between posets is also a morphism between the corresponding challenge-response relations. Because of this, we will sometimes blur the distinction between the poset $\langle P, \leq_P\rangle$ and the challenge-response relation $\langle P, P, \leq_P\rangle$. For an exposition of the theory of challenge-response relations, see [2]. Our reason for considering challenge-response relations instead of just posets is simple: finding the right challenge-response relation can help compute the cofinality of a poset.

There is also the notion of the *dual* of a challenge-response relation. That is, given $\mathcal{R} = \langle R_-, R_+, R\rangle$, the dual of $\mathcal{R}$ is the relation $\mathcal{R}^\perp = \langle R_+, R_-, \neg\tilde{R}\rangle$, where $\tilde{R}$ is the converse of R. If $\langle \phi_-, \phi_+\rangle$ is a morphism from $\mathcal{R}_1$ to $\mathcal{R}_2$, then $\langle \phi_+, \phi_-\rangle$ is a morphism from $\mathcal{R}_2^\perp$ to $\mathcal{R}_1^\perp$. If a challenge-response relation is a poset, then its bounding number equals the norm of the dual challenge-response relation.

1.3 Scales and Unbounded Chains

Some structures which help us understand posets are *scales* and *unbounded chains*:

Definition I.8. Given a poset $\mathbb{P} = \langle P, \leq_P \rangle$ and a sequence $S = \langle s_\alpha : \alpha < \kappa \rangle$ that is $\leq_P$-increasing, we call S a *scale* in $\mathbb{P}$ if

$$(\forall a \in P)(\exists \beta < \kappa)\, a \leq_P s_\beta,$$

and we call S an *unbounded chain* in $\mathbb{P}$ if

$$(\forall b \in P)(\exists \alpha < \kappa)\, s_\alpha \not\leq_P b.$$

Of course, every scale is also an unbounded chain (assuming there is no maximal element of the poset). Also, every unbounded chain has a cofinal subsequence of length a regular cardinal, and such a cofinal subsequence is also unbounded. For this reason, when we consider an arbitrary unbounded chain, we will often assume its length is a regular cardinal.

A poset $\mathbb{P}$ need not have a scale. It is straightforward to show that $\mathbb{P}$ has a scale iff the bounding number of $\mathbb{P}$ equals the cofinality of $\mathbb{P}$. On the other hand, $\mathbb{P}$ does have an unbounded chain of length the bounding number of $\mathbb{P}$ (and there are no shorter unbounded chains). In general, the set of all lengths of unbounded chains in a poset can be complicated.

When a poset $\mathbb{P}$ has an unbounded chain of length κ, there is a morphism from $\mathbb{P}$ to $\langle \kappa, \leq \rangle$:

Observation I.9. *If* $\mathbb{P} = \langle P, \leq_P \rangle$ *is a poset and* $\langle s_\alpha : \alpha < \kappa \rangle$ *is an unbounded chain* *in* $\mathbb{P}$*, then there is a morphism* $\langle \phi_-, \phi_+ \rangle$ *from* $\mathbb{P}$ *to* $\langle \kappa, \leq \rangle$*:*

$$\begin{array}{ccc} P & \leq_P & P \\ {\scriptstyle \phi_-}\uparrow & \Downarrow & \downarrow{\scriptstyle \phi_+} \\ \kappa & \leq & \kappa. \end{array}$$

Proof. Let $\phi_- : \kappa \to P$ be defined by

$$\phi_-(\alpha) = s_\alpha,$$

and let $\phi_+ : P \to \kappa$ be defined by

$$\phi_+(b) := \min\{\alpha < \kappa : s_\alpha \not\leq_P b\}. \qquad \square$$

When the unbounded chain is also a scale, there is a morphism in the opposite direction:

Observation I.10. *If* $\mathbb{P} = \langle P, \leq_P \rangle$ *is a poset and* $\langle s_\alpha : \alpha < \kappa \rangle$ *is a scale in* $\mathbb{P}$*, then there is a morphism* $\langle \psi_-, \psi_+ \rangle$ *from* $\langle \kappa, \leq \rangle$ *to* $\mathbb{P}$*:*

$$\begin{array}{ccc} \kappa & \leq & \kappa \\ {\scriptstyle \psi_-}\uparrow & \Downarrow & \downarrow{\scriptstyle \psi_+} \\ P & \leq_P & P. \end{array}$$

Proof. Let $\psi_- : P \to \kappa$ be defined by

$$\psi_-(a) := \min\{\beta < \kappa : a \leq_P s_\beta\},$$

and let $\psi_+ : \kappa \to P$ be defined by

$$\psi_+(\beta) := s_\beta. \qquad \square$$

These two observations make precise the idea that if $\mathbb{P}$ has a scale of length κ, then numerous questions about $\mathbb{P}$ can be reduced to questions about the cardinal κ. Unfortunately, the posets we will study will generally not have scales.

1.4 The Baire Hierarchy

There is a natural hierarchy on the set of Borel functions called the Baire hierarchy. Before defining this hierarchy, recall the following:

Definition I.11. A topological space is *Polish* if it has a countable dense subset and its topology is that of a complete metric space.

Examples of Polish spaces include $\mathbb{R}$ with the usual topology and ω with the discrete topology. Another important example is *Baire space*, which is the set ${}^{\omega}\omega$ of functions from ω to ω with the topology generated by the sets of the form

$$\{x \in {}^{\omega}\omega : x(0) = n_0, ..., x(k) = n_k\}$$

for some finite sequence $\langle n_0, ..., n_k \rangle$. Equivalently, the topology is induced by the metric

$$d(x,y) = \begin{cases} 2^{-\min\{n+1 : x(n) \neq y(n)\}} & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

For technical reasons, many of the results will involve Baire space instead of an arbitrary Polish space. All Polish spaces are somewhat similar to Baire space. For example, for each Polish space X, there is a continuous surjection from Baire space to X. See [30] for the precise relationship between Baire space and other Polish spaces. Our choice for focusing on Baire space is to keep the exposition simple. We may confront the fundamental issues at hand without getting sidetracked by generalities. In the few places where using Baire space as opposed to an arbitrary Polish space makes a difference, we will say so. We will now define the *Baire hierarchy*.

Definition I.12. Fix a Polish space Y. $\mathcal{B}_0(Y)$ is the set of continuous functions from ${}^{\omega}\omega$ to Y. For α satisfying $1 \leq \alpha < \omega_1$, $\mathcal{B}_\alpha(Y)$ is the set of functions which are pointwise limits of sequences of functions in $\bigcup_{\beta<\alpha} \mathcal{B}_\beta(Y)$. Functions in $\mathcal{B}_\alpha(Y)$ are called *Baire class* α. Finally, $\mathcal{B}_{\omega_1}(Y) := \bigcup_{\beta<\omega_1} \mathcal{B}_\beta(Y)$.

It is well known (see [30]) that a function $f : {}^\omega\omega \to Y$ is Borel iff $f \in \mathcal{B}_\alpha(Y)$ for some $\alpha < \omega_1$. Hence, $\mathcal{B}_{\omega_1}(Y)$ is the set of Borel functions from ${}^\omega\omega$ to Y. For each $\alpha < \omega$, there are two partially ordered sets ($\mathcal{B}_\alpha(\omega, \leq)$ and $\mathcal{B}_\alpha({}^\omega\omega, \leq^*)$) whose study will guide the results of this thesis:

Definition I.13. Given a Polish space Y and a partial ordering $\prec$ on Y, $\mathcal{B}_\alpha(Y, \prec)$ is the set $\mathcal{B}_\alpha(Y)$ ordered pointwise by $\prec$. We will denote this partial ordering by the same symbol $\prec$, so $(\forall f, g \in \mathcal{B}_\alpha(Y))$

$$f \prec g \Leftrightarrow (\forall x \in {}^\omega\omega)\, f(x) \prec g(x).$$

We make a similar definition for considering arbitrary functions:

Definition I.14. Given a set Y and a partial ordering $\prec$ on Y, $\mathrm{All}(Y, \prec)$ is the set $\mathrm{All}(Y)$ of all functions from ${}^\omega\omega$ to Y ordered pointwise by $\prec$. We denote this partial ordering by the same symbol $\prec$.

We will see that while our techniques to compute $\mathrm{cf}\,\mathcal{B}_{\omega_1}(\omega, \leq)$ can also be applied to compute $\mathrm{cf}\,\mathrm{All}(\omega, \leq)$, this is not the case when passing from $\mathrm{cf}\,\mathcal{B}_{\omega_1}({}^\omega\omega, \leq^*)$ to $\mathrm{cf}\,\mathrm{All}({}^\omega\omega, \leq^*)$.

1.5 The Results of this Thesis

The results of this thesis can be broken into two categories: combinatorial set theory and descriptive set theory. While the guiding problem is to compute $\mathrm{cf}\,\mathcal{B}_\alpha(\omega, \leq)$ and $\mathrm{cf}\,\mathcal{B}_\alpha({}^\omega\omega, \leq^*)$ for all $\alpha \leq \omega_1$, during this process it is natural to consider applications to combinatorial set theory.

1.5.1 Combinatorial Set Theory

In Chapter II we will summarize past work relevant to generalized domination. This is mostly combinatorial set theory. Starting with Chapter III, all the results are new. We will discuss the relationship between $\langle {}^{\lambda}\kappa_1, \leq \rangle$ and $\langle {}^{\lambda}\kappa_2, \leq \rangle$. This turns out to be surprisingly subtle. We will also prove that when λ is a singular strong limit cardinal and $\kappa < \lambda$, then $\mathrm{cf}\,\langle {}^{\lambda}\kappa, \leq \rangle = 2^{\lambda}$.

In Chapter V, when we develop some of our main coding theorems, we will prove the following:

Proposition I.15. *Let κ and λ be infinite cardinals. For each $A \subseteq \lambda$, there is a function $f : {}^{\kappa}\lambda \to \kappa$ such that whenever M is a transitive model of ZF satisfying ${}^{\kappa}\lambda \in M$ and some $g : {}^{\kappa}\lambda \to \kappa$ in M everywhere dominates f, then $A \in M$.*

We can remove the requirement that ${}^{\kappa}\lambda \in M$ and replace it with the requirements that $\kappa = \omega$ and $\lambda \in M$ (and therefore ${}^{<\kappa}\lambda \subseteq M$). Hence, in a certain situation, we may remove the requirement that ${}^{\kappa}\lambda \in M$, and this is very important. The proof of this special result uses the fact that well-foundedness of trees is absolute, and does not immediately generalize to the case that $\kappa > \omega$. With this special result, we obtain a surprising fact about complete Boolean algebras:

Theorem I.16. *Let λ be an infinite cardinal. Let $\mathbb{B}$ be a complete Boolean algebra. If $\mathbb{B}$ is weakly $(\lambda^{\omega}, \omega)$-distributive, then $\mathbb{B}$ is $(\lambda, 2)$-distributive.*

By $\mathbb{B}$ being weakly (μ, κ)-distributive, we mean that when forcing with $\mathbb{B}$, functions from μ to κ in the extension are everywhere dominated by functions from μ to κ in the ground model. There is a more algebraic characterization of both distributivity and weak distributivity which we will describe in Section 2.9. We may replace the component of the proof that uses the fact that well-foundedness of trees is absolute with a different absoluteness result concerning the existence of length κ paths through subtrees of ${}^{\kappa}\lambda$. We get the following variation of the theorem above:

Theorem I.17. *Let κ be a weakly compact cardinal. Let $\mathbb{B}$ be a complete Boolean algebra. If $\mathbb{B}$ is weakly $(2^\kappa, \kappa)$-distributive and is $(\alpha, 2)$-distributive for each $\alpha < \kappa$, then $\mathbb{B}$ is $(\kappa, 2)$-distributive.*

It is important that κ is weakly compact, and not just that κ has the tree property. Another variation along these lines is the following:

Theorem I.18. *Let $\mathbb{B}$ be a complete Boolean algebra. If $\mathbb{B}$ is weakly $(2^{\omega_1}, \omega_1)$-distributive, $\mathbb{B}$ is $(\omega, 2)$-distributive, and $1 \Vdash_{\mathbb{B}} (\omega_1 < \mathfrak{t})$, then $\mathbb{B}$ is $(\omega_1, 2)$-distributive.*

The cardinal $\mathfrak{t}$ is the *tower number*, which we will define in Section 5.6. The requirement that $1 \Vdash_{\mathbb{B}} (\omega_1 < \mathfrak{t})$ cannot be removed in the sense that if there exists a Suslin tree, then there is a complete Boolean algebra which is simultaneously weakly $(2^{\omega_1}, \omega_1)$-distributive and $(\omega, 2)$-distributive but is not $(\omega_1, 2)$-distributive.

1.5.2 Descriptive Set Theory

As stated before, the guiding problem of this thesis is to compute both $\mathrm{cf}\,\mathcal{B}_\alpha(\omega, \leq)$ and $\mathrm{cf}\,\mathcal{B}_\alpha({}^\omega\omega, \leq^*)$ for all $\alpha \leq \omega_1$. This will require us to develop new techniques, which we will then apply to prove some diverse and surprising results. These posets are interesting in their own right, but the original motivation for studying $\mathcal{B}_{\omega_1}({}^\omega\omega, \leq^*)$ was to provide insight into the notion of *Borel boundedness* ([3], [45]) which appears in the theory of Borel equivalence relations on ${}^\omega\omega$ all of whose equivalence classes are countable. We hope that our techniques may have applications there. Also, we chose to investigate $\mathcal{B}_\alpha({}^\omega\omega, \leq^*)$ instead of $\mathcal{B}_\alpha({}^\omega\omega, \prec)$ for some other ordering $\prec$ on ${}^\omega\omega$ because $\leq^*$ is concrete and it captures the main idea for any reasonable $\prec$. Our final result (Theorem I.27) can be viewed as applying to any reasonable $\prec$ because it applies to the weakest relation: non-equality of reals.

Observation I.19. *For each $\alpha \leq \omega_1$,*

$$\mathfrak{d} \leq \operatorname{cf} \mathcal{B}_\alpha({}^\omega\omega, \leq^*) \leq \operatorname{cf} \mathcal{B}_\alpha(\omega, \leq) \leq 2^\omega.$$

Proof. Fix $\alpha \leq \omega_1$. By mapping functions from ${}^\omega\omega$ to ${}^\omega\omega$ to their value at some fixed point, and by mapping an element of ${}^\omega\omega$ to the corresponding constant function, we easily get a morphism from $\mathcal{B}_\alpha({}^\omega\omega, \leq^*)$ to $\langle {}^\omega\omega, \leq^* \rangle$. By Observation I.4,

$$\mathfrak{d} \leq \operatorname{cf} \mathcal{B}_\alpha({}^\omega\omega, \leq^*).$$

Next, by partitioning ${}^\omega\omega$ into blocks of size ω, we see that each function in $\mathcal{B}_\alpha(\omega)$ corresponds to a function in $\mathcal{B}_\alpha({}^\omega\omega)$. It is important that this correspondence respects the levels of the Baire hierarchy, but this is routine to verify. With this correspondence, we see that there is a morphism from $\mathcal{B}_\alpha(\omega, \leq)$ to $\mathcal{B}_\alpha({}^\omega\omega, \leq)$. This implies there is a morphism from $\mathcal{B}_\alpha(\omega, \leq)$ to $\mathcal{B}_\alpha({}^\omega\omega, \leq^*)$, so by Observation I.4,

$$\operatorname{cf} \mathcal{B}_\alpha({}^\omega\omega, \leq^*) \leq \operatorname{cf} \mathcal{B}_\alpha(\omega, \leq).$$

Finally, $|\mathcal{B}_\alpha(\omega, \leq)| \leq 2^\omega$, so of course $\operatorname{cf} \mathcal{B}_\alpha(\omega, \leq) \leq 2^\omega$. □

There is no reason a priori for there to be any relationship between the cofinalities of the posets $\mathcal{B}_\alpha(\omega, \leq)$ for varying $\alpha \leq \omega_1$. The same can be said for the posets $\mathcal{B}_\alpha({}^\omega\omega, \leq^*)$ for varying $\alpha \leq \omega_1$. We will separate the discussion of the posets $\mathcal{B}_\alpha(\omega, \leq)$ from the discussion of the posets $\mathcal{B}_\alpha({}^\omega\omega, \leq^*)$.

1.5.3 Functions from ${}^{\omega}\omega$ to ω

In Chapter III, we will show that the classical proof to produce large independent families of functions can be arranged to produce Borel functions. This implies $\mathrm{cf}\,\mathcal{B}_\alpha(\omega, \leq) = 2^\omega$ for all but very small $\alpha < \omega$. However, this observation sheds no light on $\mathcal{B}_\alpha({}^{\omega}\omega, \leq^*)$.

In Chapter IV, we will show

$$\mathrm{cf}\,\mathcal{B}_0(\omega, \leq) = \mathfrak{d}.$$

This implies that an arbitrary $A \subseteq \omega$ *cannot* be encoded into a continuous function $f : {}^{\omega}\omega \to \omega$ so that A can be guessed from a dominator of f using countably many guesses. The "reason" why $\mathrm{cf}\,\mathcal{B}_0(\omega, \leq) = \mathfrak{d}$ is the following more combinatorial result, which we will prove:

Theorem I.20. *Let $\mathcal{W}$ be the set of well-founded subtrees of ${}^{<\omega}\omega$. Then*

$$\mathrm{cf}\,\langle \mathcal{W}, \subseteq \rangle = \mathfrak{d}.$$

This in turn follows from the existence of a morphism from a challenge-response relation, which will easily be seen to have norm $\mathfrak{d}$, to $\langle \mathcal{W}, \subseteq \rangle$. That morphism gives us another interesting application:

Theorem I.21. *Let M be a transitive model of ZF. Assume that*

$$(\forall f_1 \in {}^{\omega}\omega)(\exists f_2 \in {}^{\omega}\omega \cap M)\, f_1 \leq f_2.$$

Assume also that $\omega_1 = (\omega_1)^M$. Then for each well-founded tree $T_1 \subseteq {}^{<\omega}\omega$, there is some well-founded tree $T_2 \subseteq {}^{<\omega}\omega$ in M satisfying $T_1 \subseteq T_2$.

Unfortunately, to show $\mathcal{B}_0(\omega, \leq) = \mathfrak{d}$, it is important that $\mathcal{B}_0(\omega)$ is the set of continuous functions from ${}^{\omega}\omega$ to ω, as opposed to the set of continuous functions from some other Polish space X to ω.

In Chapter V, we will see a sharp transition as we pass from continuous functions to Baire class one functions. We will present a novel technique for computing $\operatorname{cf} \mathcal{B}_\alpha(\omega, \leq)$ for all $\alpha \geq 1$. The technique will have significant applications, such as the implications between weak distributivity laws for complete Boolean algebras. Computing $\operatorname{cf} \mathcal{B}_\alpha({}^\omega\omega, \leq^*)$ for $\alpha \geq 1$, on the other hand, will be of an entirely different nature. As the inclusion ordering on trees was the key to understanding continuous functions, the inclusion ordering on *clouds* turns out to be the right way to understand Baire class one functions and beyond. We will quickly develop the theory of clouds, and using them we will show that for each $\alpha \geq 1$,

$$\operatorname{cf} \mathcal{B}_\alpha(\omega, \leq) = 2^\omega.$$

We will establish this by constructing, for $\alpha \geq 1$, a morphism from $\mathcal{B}_\alpha(\omega, \leq)$ to $\langle \mathcal{P}(\omega), \leq_{\Delta^1_1} \rangle$:

$$\begin{array}{ccc} \mathcal{B}_\alpha(\omega) & \leq & \mathcal{B}_\alpha(\omega) \\ {\scriptstyle \phi_-}\uparrow & \Downarrow & \downarrow{\scriptstyle \phi_+} \\ \mathcal{P}(\omega) & \leq_{\Delta^1_1} & \mathcal{P}(\omega). \end{array}$$

The same morphism works for each $\alpha \geq 1$. By $\leq_{\Delta^1_1}$, we mean that $A \leq_{\Delta^1_1} B$ iff A is definable by a Δ^1_1 formula using B as a predicate. When $A \leq_{\Delta^1_1} B$, we say that A is hyperarithmetical in B. We use the same definition even if instead B is a type 2 object, such as a function from ${}^\omega\omega$ to ${}^\omega\omega$. We make similar definitions for other classes, such as Δ^1_2 and Σ^0_1. The following gives us the desired morphism (and much more):

Theorem I.22. *For each $A \subseteq \omega$, there is a function $f \in \mathcal{B}_1(\omega, \leq)$ such that if $g : {}^\omega\omega \to \omega$ is **any** function satisfying $(\forall x \in ({}^\omega\omega)^{L[g]})\, f(x) \leq g(x)$, then $A \leq_{\Delta^1_1} g$.*

The set A is not only Δ^1_1 definable using g as a predicate, but we can arrange f

so that there exist nodes $t_1, t_2 \in {}^{<\omega}\omega$ satisfying the following:

$$\begin{aligned} A &= \{n \in \omega : (\forall x \sqsupseteq t_1^\frown n)\, g(x) \geq |t_1^\frown n|\}, \\ \omega - A &= \{n \in \omega : (\forall x \sqsupseteq t_2^\frown n)\, g(x) \geq |t_2^\frown n|\}. \end{aligned}$$

We may view this as an *infinite coding* result. This is precisely what we described in the first section: Alice wants to send $A \subseteq \omega$ to Bob. She encodes A into the Baire class one function $f : {}^\omega\omega \to \omega$. She tries to send f to Bob, but instead an enemy steps in and substitutes a function $g : {}^\omega\omega \to \omega$ which everywhere dominates f. Given g, Bob can guess A by making countably many guesses: he simply guesses each subset of ω that is definable by some Δ^1_1 formula which uses g as a predicate.

We discuss two encoding techniques: *horizontal coding* and *vertical coding.* The theorem above can be proved using either one. We will see that the two techniques have different useful generalizations, so we must study both. We will analyze exactly how sloppy we can be to still perform vertical coding. The following is an example of that analysis:

Proposition I.23. *Let $a \in \mathbb{R}$ be a real. Let $f : \mathbb{R} \to \mathbb{R}$ be the function*

$$f(x) := \begin{cases} \frac{1}{x-a} & \text{if } x \neq a, \\ 0 & \text{if } x = a. \end{cases}$$

If $g : \mathbb{R} \to \mathbb{R}$ is a function which everywhere dominates f, then $a \in L[g]$. Hence, if g is also Borel, then $a \in L[c]$ where c is any Borel code for g.

In this proposition, the relation "$a \in L[g]$" is replacing the "$A \leq_{\Delta^1_1} g$" of the theorem above, but this is not essential. Using horizontal coding, we will prove the following:

Proposition I.24. *For each $A \subseteq {}^\omega\omega$, there is a function $f : {}^\omega\omega \to \omega$ such that whenever $g : {}^\omega\omega \to \omega$ is* ***any*** *function satisfying $f \leq g$, then A is $\mathbf{\Delta}^1_1$ in g.*

By $\mathbf{\Delta}^1_1$, we mean definable by a Δ^1_1 formula using g as a predicate and some real as a parameter. The proposition implies the following:

Corollary I.25. *For each $A \subseteq \mathbb{R}$, there is a function $f : \mathbb{R} \to \omega$ such that if $g : \mathbb{R} \to \omega$ satisfies $f \leq g$, then $A \in L(\mathbb{R}, g)$.*

1.5.4 Functions from ${}^\omega\omega$ to ${}^\omega\omega$

In Chapter VI, we will discuss various obstructions to computing the cofinality of $\mathcal{B}_\alpha({}^\omega\omega, \leq^*)$ for $\alpha \geq 1$. We also establish various limits to what kinds of infinite coding theorems can exist. First, we show that if we consider the poset $\mathrm{All}({}^\omega\omega, \leq^*)$ of *all* functions from ${}^\omega\omega$ to ${}^\omega\omega$ (instead of just the Borel ones) ordered by pointwise eventual domination, then there is no way in ZFC to prove that an arbitrary subset of $\mathbb{R}$ can be encoded into one of these functions. This contrasts with the situation with $\mathrm{All}(\omega, \leq)$, because a result like Corollary I.25 shows that encoding of arbitrary subsets of $\mathbb{R}$ into that poset *is* possible. In essence, the problem with $\mathrm{All}({}^\omega\omega, \leq^*)$ is that there might exist a scale in $\langle {}^\omega\omega, \leq^* \rangle$ of length 2^ω. A scale, however, is an object whose existence requires some amount of the axiom of choice, and it is not relevant when we investigate $\mathcal{B}_\alpha({}^\omega\omega, \leq^*)$ for $\alpha \leq \omega_1$.

Next in Chapter VI, we will establish that some naive attempts using vertical coding to show $\mathrm{cf}\, \mathcal{B}_\alpha({}^\omega\omega, \leq^*) = 2^\omega$ (for $\alpha \geq 1$) fail. To prove the failure of the techniques, we will use Sacks forcing. Our reason for spending the energy to do this is because we want to be sure we have the simplest encoding scheme possible. As we will see in Chapter VII, an encoding scheme *does* exist, but the proof that it works is very complicated and was time consuming to discover. We do not want the readers to waste time exploring paths on their own that we know lead to dead ends.

Next in Chapter VI, we observe that if we consider projective (instead of just Borel) functions from ${}^{\omega}\omega$ to ${}^{\omega}\omega$ ordered by pointwise eventual domination, then arbitrary subsets of ω *cannot* be encoded into these functions (in a canonical way) assuming the following: 1) there is a projective well-ordering of ${}^{\omega}\omega$, and 2) $\omega_2 \leq \mathfrak{b}$. Since it is consistent with ZFC that these conditions may be satisfied simultaneously, we have that ZFC cannot prove an infinite coding theorem for projective functions from ${}^{\omega}\omega$ to ${}^{\omega}\omega$. This leaves open the question of whether further natural axioms (for example, the axiom of projective determinacy) imply a coding theorem for projective functions.

In Chapter VII, we establish that for each $\alpha \geq 1$,

$$\operatorname{cf} \mathcal{B}_\alpha({}^{\omega}\omega, \leq^*) = 2^\omega.$$

We start the chapter by illustrating what was lacking from the naive vertical coding attempt of the previous chapter. We then present a proof that

$$\operatorname{cf} \mathcal{B}_1({}^{\omega}\omega, \leq^*) = 2^\omega$$

using techniques entirely different from those in Chapter V. However, still as before, we will prove this by constructing a morphism from $\mathcal{B}_1({}^{\omega}\omega, \leq^*)$ to the relation $\langle \mathcal{P}(\omega), \leq_{\Delta^1_1} \rangle$:

$$\begin{array}{ccccc} \mathcal{B}_1({}^{\omega}\omega) & & \leq^* & & \mathcal{B}_1({}^{\omega}\omega) \\ \uparrow & & \Downarrow & & \downarrow \\ \mathcal{P}(\omega) & & \leq_{\Delta^1_1} & & \mathcal{P}(\omega). \end{array}$$

Next, the challenge becomes to show that

$$\operatorname{cf} \mathcal{B}_2({}^{\omega}\omega, \leq^*) = 2^\omega.$$

Ultimately, this requires us to clarify the argument for $\mathcal{B}_1({}^\omega\omega, \leq^*)$ and develop a more powerful technique. We isolate the right statements to prove using induction to handle $\mathcal{B}_\alpha({}^\omega\omega, \leq^*)$ for $\alpha < \omega_1$. This will give us, for each α satisfying $1 \leq \alpha < \omega_1$, a morphism from $\mathcal{B}_\alpha({}^\omega\omega, \leq^*)$ to $\langle \mathcal{P}(\omega), \leq_{\Delta^1_2} \rangle$:

$$\begin{array}{ccc} \mathcal{B}_\alpha({}^\omega\omega) & \leq^* & \mathcal{B}_\alpha({}^\omega\omega) \\ \uparrow & \Downarrow & \downarrow \\ \mathcal{P}(\omega) & \leq_{\Delta^1_2} & \mathcal{P}(\omega). \end{array}$$

Indeed, it suffices to construct the following morphism:

$$\begin{array}{ccc} \mathcal{B}_1({}^\omega\omega) & \leq^* & \mathcal{B}_{\omega_1}({}^\omega\omega) \\ \uparrow & \Downarrow & \downarrow \\ \mathcal{P}(\omega) & \leq_{\Delta^1_2} & \mathcal{P}(\omega). \end{array}$$

The existence of this follows from the next theorem. The reason for Δ^1_2 is because of the complexity of the graph of the function Ψ used in the proof:

Theorem I.26 (Borel Dominator Δ^1_2 Coding Theorem). *For each $A \subseteq \omega$, there is a Baire class one function $f : {}^\omega\omega \to {}^\omega\omega$ such that whenever $g : {}^\omega\omega \to {}^\omega\omega$ is a Borel function satisfying*

$$(\forall x \in {}^\omega\omega)(\exists c \in \omega)\, f(x)(c) \leq g(x)(c),$$

then A is Δ^1_2 in any code for g.

We have now completed our quest to compute $\mathcal{B}_\alpha(\omega, \leq)$ and $\mathcal{B}_\alpha({}^\omega\omega, \leq^*)$ for all $\alpha \leq \omega_1$. We can now justify that our choice of considering $\leq^*$ instead of some other relation on ${}^\omega\omega$ did not matter. The theorem above involves the relation

$$(\exists c \in \omega)\, f(x)(c) \leq g(x)(c)$$

between $f(x)$ and $g(x)$. The proof generalizes easily to handle any reasonable relation R between $f(x)$ and $g(x)$. Specifically, all we need is for $R \subseteq {}^\omega\omega \times {}^\omega\omega$ to be any relation such that there exists a continuous function $j : {}^\omega\omega \to {}^\omega\omega$ satisfying

$$(\forall y \in {}^\omega\omega)\, \neg j(y) R y.$$

Fixing such an R, the generalization may be stated as follows: for each $A \subseteq \omega$, there is a Baire class one function $f : {}^\omega\omega \to {}^\omega\omega$ such that whenever $g : {}^\omega\omega \to {}^\omega\omega$ is a Borel function satisfying

$$(\forall x \in {}^\omega\omega)\, f(x) R g(x),$$

then A is Δ^1_2 in any code for g.

Essentially all relations studied in the area of cardinal characteristics of the continuum (are equivalent to relations which) satisfy this hypothesis. There is a weakest relation out of all these: non-equality. We now have a remarkably strong result with an analysis flavor. We can use an arbitrary Polish space X instead of ${}^\omega\omega$, at the cost of perhaps slightly increasing the complexity of f:

Theorem I.27. *Let X and Y be Polish spaces with X uncountable. For each $A \subseteq \omega$, there is a Borel $f : X \to Y$ such that whenever $g : X \to Y$ is Borel, then at least one of the following holds:*

1) $(\exists x \in X)\, f(x) = g(x)$;

2) A is Δ^1_2 in any code for g.

The strength of this result is a testament to the underlying method. The development of the method is by far the deepest contribution of this thesis.

We leave the reader with a puzzling question: can Theorem I.27 be generalized to work when g is a projective function? By the observation that there can exist a long projective well-ordering of the reals while simultaneously $\omega_2 \leq \mathfrak{b}$, we cannot expect ZFC to prove such a generalization. We may ask whether it follows from projective determinacy or the existence of large cardinals. If so, this would likely require inventing a different proof of Theorem I.27, which is no easy task.

Finally, in the appendix we will present a few lemmas about Sacks forcing which we use. We will also present several ideas which, although they were not used in this thesis, are still natural for tackling problems in the area of cardinal characteristics.

1.6 Notation

In addition to what we have defined in this introduction, within this section we will fix the rest of the notation for this thesis. With very few exceptions, we will use standard set theoretic notation and terminology. When we say cardinal, we will always mean infinite cardinal. By antichain, we mean strong antichain (elements are pairwise incompatible). We write $a \perp b$ when a and b are incompatible. The reader should have basic familiarity with forcing, including nice names. Given two sets X and Y, $X \sqcup Y$ is the disjoint union of X and Y. Given a set X and a cardinal κ, $[X]^\kappa$ is the collection of size κ subsets of X, and $[X]^{<\kappa}$ is the collection of size $< \kappa$ subsets of X. By λ-tree, we mean a tree all of whose levels have size $< \lambda$. By $\mu \to (\kappa)^n_\nu$, we mean the standard partition relation (given any coloring of $[\kappa]^n$ using ν colors, there is a homogeneous subset of κ of size μ). By MA, we mean Martin's axiom (the version consistent with CH).

When we say that $A \subseteq \omega$ is Π^1_1 in a set B, we mean that membership in A is determined by a Π^1_1 formula which uses B as a predicate. We say that A is Δ^1_1 in B if both A and $\omega - A$ are Π^1_1 in B. We use a similar definition for A being Δ^1_2 in B. By $\leq_T$, we mean Turing reducibility. These are the only recursion theoretic definitions we will need.

We will use a number of concepts from descriptive set theory. We will use *codes* for Borel and projective sets. The theory of such codes can be found in [30] and [39]. The point is that Borel sets, and more generally projective sets, can be coded by individual real numbers, and properties of the sets can be reduced to properties of the reals which code them. From a real which codes a Borel set, the process by which the set is built up in the Borel hierarchy may be recovered. By AD we mean the axiom of determinacy. Θ is the smallest ordinal which $\mathbb{R}$ cannot be surjected onto. We use w.s. as an abbriviation for winning strategy.

Whenever we say cardinal, we shall mean infinite cardinal. Given sets A and B, let ${}^{A}B$ denote the set of functions from A to B. As usual, given a function $f : X \to Y$, we write $\mathrm{Dom}(f) = X$ for the domain of f, $\mathrm{Im}(f) \subseteq Y$ for the image of f, and given $S \subseteq X$, $f \restriction S$ is the restriction of f to S. Given $S \subseteq \mathrm{Dom}(f)$, we write $f``(S)$ for $\mathrm{Im}(f \restriction S)$. Given an expression $e(x)$ which depends on x, we write

$$x \mapsto e(x)$$

for the function which given x returns $e(x)$. By a *sequence*, we mean a function whose domain is an ordinal. The expression $\langle a, b, c\rangle$ denotes the sequence which maps 0 to a, 1 to b, and 2 to c. Given an ordinal κ and a set X, let ${}^{<\kappa}X$ be the collection of all functions whose domain is a proper initial segment of κ:

$${}^{<\kappa}X := \bigcup_{\alpha<\kappa} {}^{\alpha}X.$$

Given two sequences t and s, we write $t \sqsubseteq s$ if s is an end-extension of t. That is,

$$t \sqsubseteq s \text{ iff } s \restriction \mathrm{Dom}(t) = t.$$

Given two sequences t and s, we write $t^\frown s$ for the concatenation of t and s. Given $t \in {}^{<\kappa}X$ and $a \in X$, we may abuse notation and write $t^\frown a$ when we mean $t^\frown\langle a\rangle$.

A set $T \subseteq {}^{<\kappa}X$ is a *tree* if it is closed under taking initial segments. Elements of T we generally call *nodes*. We call $\emptyset$ the *root* of T (assuming T is non-empty). Nodes which have no proper end-extensions in T we call leaf nodes. We write $[T] \subseteq {}^{\kappa}X$ for the set of all length κ paths through T:

$$\{x \in {}^{\kappa}X : (\forall \alpha < \kappa)\, x \upharpoonright \alpha \in T\}.$$

Of course, this definition depends on κ, but it will always be clear from context what we mean. Given $t \in {}^{<\omega}\omega$, we write $[t]$ for the set of all $x \in {}^{\omega}\omega$ satisfying $x \sqsupseteq t$. Given $x \in {}^{\omega}\omega$, we write $[[x]]$ for the set of $t \in {}^{<\omega}\omega$ satisfying $t \sqsubseteq x$ (this is not standard notation). Given $\alpha < \kappa$, the α-th level of T is the set

$$T \cap {}^{\alpha}X.$$

The *height* of T is

$$\sup\{\alpha < \kappa : T \cap {}^{\alpha}X \neq \emptyset\}.$$

Given $t \in T$, we define

$$\mathrm{Succ}_T(t) := \{a \in X : t^\frown a \in T\}.$$

If $\kappa = \omega$, we say that T is well-founded if it has no infinite paths. If T is well-founded then to each $t \in T$ we may assign a rank $\mathrm{rank}(T,t)$ as follows:

$$\mathrm{rank}(T,t) := \begin{cases} 1 & \text{if } t \text{ is a leaf node of } T, \\ \sup\{\mathrm{rank}(T, t^\frown a) + 1 : a \in \mathrm{Succ}_T(t)\} & \text{otherwise.} \end{cases}$$

Note that we are using the convention that leaf nodes of T have rank 1, which allows us define the rank of those $t \in {}^{<\omega}X$ not in T to be 0. The rank of the tree T itself we define to be the rank of the root:

$$\mathrm{rank}(T) := \mathrm{rank}(T, \emptyset).$$

The following definitions will help us define functions which are difficult to everywhere dominate.

Definition I.28. Let X be a set and κ be a cardinal. Let $T \subseteq {}^{<\kappa}X$ be a tree. The function $\mathrm{Exit}(T) : {}^{\kappa}X \to \kappa$ is defined by

$$\mathrm{Exit}(T)(x) := \begin{cases} 0 & \text{if } x \in [T], \\ \min\{\alpha : x \restriction \alpha \notin T\} & \text{otherwise.} \end{cases}$$

That is, $\mathrm{Exit}(T)(x)$ is the level at which x *exits* the tree T (and is 0 if x does not exit the tree). A more general definition is the following:

Definition I.29. Let X be a set and κ be a cardinal. Let $C \subseteq {}^{<\kappa}X$ be such that for each $x \in {}^{\kappa}X$, $\{\alpha < \kappa : x \restriction \alpha \in C\}$ is bounded below κ. The function $\mathrm{Rep}(C) : {}^{\kappa}X \to \kappa$ is defined by

$$\mathrm{Rep}(C)(x) := \sup\{\alpha : x \restriction \alpha \in C\}.$$

A set $C \subseteq {}^{<\kappa}X$ which satisfies the hypothesis of this definition we call a *cloud*. This definition allows us to define more functions than the previous one because given a tree $T \subseteq {}^{<\kappa}X$, the set C of sequences just outside the tree forms a cloud and $\mathrm{Exit}(T) = \mathrm{Rep}(C)$. The set of all initial segments of elements of a cloud need not be a cloud. We will generally be concerned with clouds in the case that $\kappa = \omega$. If $T \subseteq {}^{<\kappa}X$ is a tree with no length κ branches, then T is a cloud. The abbreviation "Rep" stands for *representation*.

CHAPTER II

Past Work

The purpose of this chapter is to summarize relevant past work on the problem of understanding the cofinality of $\langle {}^{\lambda}\kappa, \leq \rangle$, and generalized domination in general. The reader may skip this chapter without loss of continuity. On the other hand, the reader interested in $\langle {}^{\lambda}\kappa, \leq \rangle$ but not $\mathcal{B}_{\alpha}(\omega, \leq)$ or $\mathcal{B}_{\alpha}({}^{\omega}\omega, \leq^*)$ for $\alpha \leq \omega_1$ will enjoy this self contained chapter. There are many statements that have implications for the cofinality of $\langle {}^{\lambda}\kappa, \leq \rangle$ scattered throughout the literature. We have collected and organized them together.

To compute $\operatorname{cf} \mathcal{B}_{\alpha}(\omega, \leq)$ and $\operatorname{cf} \mathcal{B}_{\alpha}({}^{\omega}\omega, \leq^*)$, one would first look to the "usual techniques". We feel obligated to collect a list of these, even though they do not solve our problem. Most of them belong to what may be called *uncountable* infinitary combinatorics (in contrast to those combinatorial questions about the continuum which are of a *countable* nature). Also, our approach for computing $\operatorname{cf} \mathcal{B}_{\alpha}(\omega, \leq)$ and $\operatorname{cf} \mathcal{B}_{\alpha}({}^{\omega}\omega, \leq^*)$ is to prove theorems about encoding and decoding, which is quite different from most of these combinatorial methods.

We begin by describing the simplest ways to show $\operatorname{cf} \langle {}^{\lambda}\kappa, \leq \rangle$ is large. The *unbounded subset bound* is still used by the more subtle methods. Next, we explain why we are studying $\langle {}^{\lambda}\kappa, \leq \rangle$ instead of $\langle {}^{\lambda}\kappa, \leq^* \rangle$, and point out that their cofinalities are equal. We then describe some of what is known about $\langle {}^\lambda\lambda, \leq^* \rangle$. Next, we change gears slightly to summarize work on posets in descriptive set theory similar to $\mathcal{B}_\alpha(\omega, \leq)$ and $\mathcal{B}_\alpha({}^\omega\omega, \leq^*)$.

We then return to infinitary combinatorics, and first summarize the implications for $\mathrm{cf}\,\langle {}^\lambda\kappa, \leq \rangle$ when 2^ω is a real-valued measurable cardinal. We then discuss one of the most important problems related to computing $\mathrm{cf}\,\langle {}^\lambda\kappa, \leq \rangle$: constructing large $\mathcal{I}$-almost disjoint families for some κ^+-complete ideal $\mathcal{I}$ on λ. There are various techniques for creating new families from old ones, which we have organized together. Next, we discuss a problem whose importance rivals the construction of large $\mathcal{I}$-disjoint families: the construction of large κ^+-independent families. From this, we get that $\lambda^\kappa = \lambda$ implies $\mathrm{cf}\,\langle {}^\lambda\kappa, \leq \rangle = 2^\lambda$.

At the end of the chapter, we show a connection between everywhere domination and finding paths through trees. This illustrates the essential idea behind the Jockusch [28] and Solovay [42] result that Δ^1_1 subsets of ω can be encoded into $\langle {}^\omega\omega, \leq \rangle$. Finally, we show the important connection to weak distributivity laws for complete Boolean algebras.

2.1 Basics

Given a cardinal λ and a regular cardinal $\kappa \leq \lambda$, we will review the basic ways to show that $\mathrm{cf}\,\langle {}^\lambda\kappa, \leq \rangle$ is large. These are different from the techniques we will develop to "encode information" into functions which can then be "decoded" from dominators of those functions.

Proposition II.1 (Standard Diagonalization Bound)**.** *For any regular cardinal κ and any cardinal $\lambda \geq \kappa$,* $\mathrm{cf}\,\langle {}^\lambda\kappa, \leq \rangle \geq \lambda^+$.

Proof. Consider any $\mathcal{G} = \{g_\alpha \in {}^\lambda\kappa : \alpha < \lambda\}$ of size at most λ. Define $f \in {}^\lambda\kappa$ by

$$f(\alpha) := g_\alpha(\alpha) + 1.$$

Then f is not everywhere dominated by any member of $\mathcal{G}$, so $\mathcal{G}$ is not cofinal. □

Indeed, this proof can be easily modified to show $\mathrm{cf}\,\langle {}^\lambda\kappa, \leq^* \rangle \geq \lambda^+$, but we will wait until the next section to discuss $\langle {}^\lambda\kappa, \leq^* \rangle$. This argument is atypical in that we start with an alleged dominating family, and then we use this to create a novel function to get a contradiction. This contrasts with the approach of first building a large family of functions all of whose subsets of a certain size are unbounded, and then appealing to the pigeon hole principle to select one of these subsets. We will describe this approach now. First, recall the following.

Proposition II.2 (Infinite Pigeon Hole Principle)**.** *Let μ be an infinite cardinal and suppose it is partitioned into pieces.*

1) If there are $< \mathrm{cf}(\mu)$ pieces, then there is a piece with μ elements.

2) If there are $< \mu$ pieces, then for each $\mu' < \mu$ there is a piece with more than μ' elements.

Proposition II.3 (Unbounded Subset Bound)**.** *Let μ be an infinite cardinal and $\mathbb{P} = \langle X, \leq \rangle$ be a poset. Suppose $\mathcal{F} \subseteq X$ and all size μ subsets of $\mathcal{F}$ are unbounded in $\mathbb{P}$ (and $\mu \leq |\mathcal{F}|$). Assume one of the following:*

1) $\mu < |\mathcal{F}|$;

2) $\mu = |\mathcal{F}|$ and μ is regular.

Then $\mathcal{F}$ cannot be dominated by $< |\mathcal{F}|$ elements of X. Hence,

$$\mathrm{cf}\,\mathbb{P} \geq |\mathcal{F}|.$$

Proof. Let $\mathcal{G} \subseteq X$ have size $< |\mathcal{F}|$. Suppose, towards a contradiction, that

$$(\forall f \in \mathcal{F})(\exists g \in \mathcal{G})\, f \leq g.$$

Partition $\mathcal{F}$ into $|\mathcal{G}|$ pieces, where all elements of a piece are below a single element of $\mathcal{G}$. Since we are assuming either 1) or 2), by the infinite pigeon hole principle, there is a single piece with at least μ elements. That is, there are μ elements of $\mathcal{F}$ all below a single element of $\mathcal{G}$. This is a contradiction, because we assumed each size μ subset of $\mathcal{F}$ is unbounded in $\mathbb{P}$. □

Apparently all classical ways to show that cf $\langle {}^{\lambda}\kappa, \leq \rangle$ is large use this bound. Often the arguments use $\mu = \kappa$. However, in the next chapter when we prove cf $\langle {}^{\lambda}\kappa, \leq \rangle = 2^{\lambda}$ for λ a singular strong limit cardinal and $\kappa < \lambda$, we will see that it is useful for μ to satisfy the partition relation

$$\mu \to (\kappa)^2_{\mathrm{cf}(\lambda)}.$$

Note the requirement that *all* size μ subsets of $\mathcal{F}$ are unbounded can be weakened to almost all with respect to a sufficiently complete ideal on $\mathcal{F}$. We will not need this generalization, but the interested reader may find it useful. We say that an ideal $\mathcal{I}$ is κ-complete if unions of $< \kappa$ sets in $\mathcal{I}$ are in $\mathcal{I}$. Also, given an ideal $\mathcal{I}$, the set $\mathcal{I}^+$ is the collection of subsets of the underlying set not in $\mathcal{I}$. The κ-completeness of an ideal can be viewed as a pigeon hole principle:

Proposition II.4 (Idealized Infinite Pigeon Hole Principle)**.** *If μ is an infinite cardinal, $\mathcal{I}$ is a κ-complete ideal on μ, and μ is partitioned into $< \kappa$ pieces, then one of the pieces is in $\mathcal{I}^+$.*

Proposition II.5 (Idealized Unbounded Subset Bound)**.** *Let $\mathbb{P} = \langle X, \leq \rangle$ be a poset. Let $\mathcal{F} \subseteq X$ be infinite and let $\mathcal{I}$ be a κ-complete ideal on $\mathcal{F}$. Suppose all subsets of*

$\mathcal{F}$ in $\mathcal{I}^+$ are unbounded in $\mathbb{P}$. Then $\mathcal{F}$ cannot be dominated by $< \kappa$ elements of X. Hence, $\operatorname{cf} \mathbb{P} \geq \kappa$.

Proof. The proof is almost identical to that of Proposition II.3, except we use the idealized infinite pigeon hole principle. □

2.2 Everywhere vs. Eventual Domination

Let $\kappa \leq \lambda$ be infinite cardinals with κ regular. In the literature, the poset $\langle {}^{\lambda}\kappa, \leq^* \rangle$ of functions from λ to κ ordered by *eventual domination* is studied more than $\langle {}^{\lambda}\kappa, \leq \rangle$. We say g eventually dominates f, and write $f \leq^* g$, precisely when

$$\{x \in \lambda : f(x) > g(x)\} \tag{2.1}$$

is bounded below λ. In general, for any ideal $\mathcal{I}$ on λ, $f \leq_{\mathcal{I}} g$ iff the set (2.1) is in $\mathcal{I}$. More generally, given any product of regular cardinals $\prod_{\alpha<\lambda} \kappa_\alpha$ (treating κ_α as the poset $\langle \kappa_\alpha, \leq \rangle$) and any ideal $\mathcal{I}$ on λ, we can consider the poset

$$\langle \prod_{\alpha<\lambda} \kappa_\alpha, \leq_{\mathcal{I}} \rangle$$

defined in the expected way. The problem of understanding the cofinality of these posets is extremely broad. Indeed, it encompasses PCF theory and ultrapowers of ω. Because of the breadth of this problem, we need to restrict our attention to specific cases to make progress. For further information on $\langle {}^{\lambda}\kappa, \leq_{\mathcal{I}} \rangle$ and even more general posets, see [37]. We will now explain why we are investigating everywhere domination.

First, everywhere domination serves as a natural boundary for the general problem. That is, for any ideal $\mathcal{I}$ on λ, there is a (trivial) morphism from $\langle \prod_{\alpha<\lambda} \kappa_\alpha, \leq \rangle$

to $\langle\prod_{\alpha<\lambda}\kappa_\alpha, \leq_{\mathcal{I}}\rangle$. Hence, this is the "top layer" of the hierarchy of these posets. This layer also has internal structure. For example, given a sequence $\langle\kappa_\alpha : \alpha < \lambda_2\rangle$ of regular cardinals and $\lambda_1 \leq \lambda_2$, there is a (trivial) morphism from $\langle\prod_{\alpha<\lambda_2}\kappa_\alpha, \leq\rangle$ to $\langle\prod_{\alpha<\lambda_1}\kappa_\alpha, \leq\rangle$. In particular, for infinite cardinals $\kappa \leq \lambda_1 \leq \lambda_2$, there is a morphism from $\langle{}^{\lambda_2}\kappa, \leq\rangle$ to $\langle{}^{\lambda_1}\kappa, \leq\rangle$. In the next chapter in Section 3.3, we will show there is more subtle structure. For example, if λ is an infinite cardinal and $\kappa_1 \leq \kappa_2 \leq \lambda$ are regular cardinals satisfying $\kappa_2^{\kappa_1} \leq \lambda$, then there is a morphism from $\langle{}^{\lambda}\kappa_1, \leq\rangle$ to $\langle{}^{\lambda}\kappa_2, \leq\rangle$.

Since everywhere domination is at the top of the hierarchy, it is the natural relation to attempt to "encode information into". For example, if $\Gamma \subseteq \mathcal{P}(\omega)$ and $\leq_L$ is the constructibility ordering, then if there is a morphism from $\langle{}^{\omega_1}\omega, \leq_{\mathcal{I}}\rangle$ to $\langle\Gamma, \leq_L\rangle$ for some ideal $\mathcal{I}$ on ω_1, then there is one when $\mathcal{I} = \{\emptyset\}$. Since we want to prove that these kinds of morphisms do exist, posets of the form $\langle{}^{\lambda}\kappa, \leq_{\mathcal{I}}\rangle$ for $\mathcal{I} = \{\emptyset\}$ are the appropriate candidates to investigate.

However, since eventual domination is studied much more than everywhere domination, we will explain how they are related. First, note that the standard diagonalization bound from the previous section easily extends to eventual domination:

Proposition II.6 (Standard Diagonalization Bound)**.** *If κ is a regular cardinal and $\lambda \geq \kappa$ is a cardinal, then* $\operatorname{cf}\langle{}^{\lambda}\kappa, \leq^*\rangle \geq \lambda^+$.

Proof. Let $\{X_\alpha : \alpha < \lambda\}$ be a partition of λ into sets of size λ. Consider any $\mathcal{G} = \{g_\alpha \in {}^{\lambda}\kappa : \alpha < \lambda\}$. Define $f \in {}^{\lambda}\kappa$ such that

$$(\forall\alpha \in \lambda)(\forall x \in X_\alpha)\, f(x) = g_\alpha(x) + 1.$$

Then f is not eventually dominated by any member of $\mathcal{G}$. □

Indeed, the same argument shows that whenever $\mathcal{I}$ is an ideal on λ such that λ

can be partitioned into λ sets X_α each in $\mathcal{I}^+$, then $\mathrm{cf}\,\langle{}^\lambda\kappa, \leq_\mathcal{I}\rangle \geq \lambda^+$.

Now, of course there is a morphism from $\langle{}^\lambda\kappa, \leq\rangle$ to $\langle{}^\lambda\kappa, \leq^*\rangle$. Even though a morphism need not exist in the opposite direction, it turns out that the posets have the same cofinality. First, note the following:

Lemma II.7. *For any $\kappa \leq \lambda$,*

$$\mathrm{cf}\,\langle{}^\lambda\kappa, \leq\rangle = \mathrm{cf}\,\langle{}^\lambda\kappa, \leq^*\rangle \cdot \sum_{x<\lambda} \mathrm{cf}\,\langle{}^x\kappa, \leq\rangle.$$

Proof. The $\geq$ direction is easy. For the other direction, let $\mathcal{F} \subseteq {}^\lambda\kappa$ be cofinal in $\langle{}^\lambda\kappa, \leq^*\rangle$ having minimal cardinality. For each $x < \lambda$, let $\mathcal{H}_x \subseteq {}^x\kappa$ be cofinal in $\langle{}^x\kappa, \leq\rangle$ having minimal cardinality. For each $f \in \mathcal{F}$, $x < \lambda$, and $h \in \mathcal{H}_x$, define $g_{f,x,h} \in {}^\lambda\kappa$ by

$$g_{f,x,h}(\alpha) := \begin{cases} h(\alpha) & \text{if } \alpha < x, \\ f(\alpha) & \text{otherwise.} \end{cases}$$

The family $\{g_{f,x,h} : f \in \mathcal{F} \wedge x < \lambda \wedge h \in \mathcal{H}_x\}$ is cofinal in $\langle{}^\lambda\kappa, \leq\rangle$ and has size

$$|\mathcal{F}| \cdot \sum_{x<\lambda} |\mathcal{H}_x|,$$

so we are done. □

The idea in this proof is present in the proof that when A is a progressive set ($|A| < \min A$) of regular cardinals, $\max \mathrm{pcf}(A) = \mathrm{cf}\,\langle\prod A, \leq\rangle$ ([23] Theorem 3.4.21). The relevant part of the argument is the (easily verifiable) fact that given ideals $\mathcal{I}_1 \subseteq \mathcal{I}_2$ on a cardinal λ and any sequence $\langle\kappa_\alpha : \alpha < \lambda\rangle$ of regular cardinals,

$$\mathrm{cf}\,\langle\prod_{\alpha<\lambda} \kappa_\alpha, \leq_{\mathcal{I}_1}\rangle \leq \mathrm{cf}\,\langle\prod_{\alpha<\lambda} \kappa_\alpha, \leq_{\mathcal{I}_2}\rangle \cdot \sum_{X\in\mathcal{I}_2} \mathrm{cf}\,\langle\prod_{\alpha\in X} \kappa_\alpha, \leq_{\mathcal{I}_1}\rangle.$$

This is an inequality instead of an equality because we have a sum of possibly 2^λ terms on the right hand side. Here is the other trick:

Lemma II.8. *Let $\kappa, \lambda_1, \lambda_2$ be infinite cardinals with $\lambda_1 < \lambda_2$. Then*

$$\mathrm{cf}\,\langle {}^{\lambda_1}\kappa, \leq \rangle \leq \mathrm{cf}\,\langle {}^{\lambda_2}\kappa, \leq^* \rangle.$$

Proof. For each $f \in {}^{\lambda_1}\kappa$, let $f' \in {}^{\lambda_2}\kappa$ be the function defined by

$$f'(\lambda_1 \cdot \alpha + \beta) := f(\beta)$$

for $\alpha < \lambda_2$ and $\beta < \lambda_1$. That is, f' is the function f repeated λ_2 times. Let $\mathcal{G} \subseteq {}^{\lambda_2}\kappa$ be cofinal in $\langle {}^{\lambda_2}\kappa, \leq^* \rangle$. For each $g \in \mathcal{G}$ and $\alpha < \lambda_2$, let $g_\alpha \in {}^{\lambda_1}\kappa$ be the function

$$g_\alpha(\beta) := g(\lambda_1 \cdot \alpha + \beta).$$

Now, if $f' \leq^* g$, then $(\exists \alpha < \lambda_2)\, f \leq g_\alpha$. Thus, $\{g_\alpha : g \in \mathcal{G} \wedge \alpha < \lambda_2\}$ is cofinal in $\langle {}^{\lambda_1}\kappa, \leq \rangle$ and has size $|\mathcal{G}|$. $\square$

Corollary II.9. *For any $\kappa \leq \lambda$,*

$$\mathrm{cf}\,\langle {}^{\lambda}\kappa, \leq \rangle = \mathrm{cf}\,\langle {}^{\lambda}\kappa, \leq^* \rangle.$$

Proof. By the preceding two lemmas,

$$\begin{aligned}
\mathrm{cf}\,\langle {}^{\lambda}\kappa, \leq^* \rangle &\leq \mathrm{cf}\,\langle {}^{\lambda}\kappa, \leq \rangle \\
&= \mathrm{cf}\,\langle {}^{\lambda}\kappa, \leq^* \rangle \cdot \sum_{x<\lambda} \mathrm{cf}\,\langle {}^{x}\kappa, \leq \rangle \\
&\leq \mathrm{cf}\,\langle {}^{\lambda}\kappa, \leq^* \rangle \cdot \sum_{x<\lambda} \mathrm{cf}\,\langle {}^{\lambda}\kappa, \leq^* \rangle \\
&= \mathrm{cf}\,\langle {}^{\lambda}\kappa, \leq^* \rangle.
\end{aligned}$$

This chain of inequalities gives us the desired equivalence. $\square$

2.3 Functions from λ to λ

Instead of studying $\langle {}^\lambda\kappa, \leq^* \rangle$ in general, one usually studies $\langle {}^\lambda\lambda, \leq^* \rangle$ (assuming λ is regular). Moreover, usually $\lambda = \omega$. The poset $\langle {}^\omega\omega, \leq^* \rangle$ is the one most likely to appear in applications to other branches of mathematics. In the study of the set theory of the real line, $\langle {}^\omega\omega, \leq^* \rangle$ is near the center of a complicated interconnected plethora of structures, which taken together we may call *the continuum.* It is also highly chaotic in the sense that we can force its cofinality and bounding number to be almost anything we want (subject to the constraints given by its interconnections to the rest of the structures of the continuum).

Hechler [21] has shown that given a poset $\mathbb{Q}$ in which every countable subset has an upper bound, there is a c.c.c. forcing $\mathbb{H}$ which forces a strictly order-preserving cofinal embedding of $\mathbb{Q}$ into $\langle {}^\omega\omega, \leq^* \rangle$. Now, let λ be a regular cardinal. To be concise, let us write $\mathfrak{b}(\lambda)$ for $\mathfrak{b}\,\langle {}^\lambda\lambda, \leq^* \rangle$ and $\mathfrak{d}(\lambda)$ for $\mathrm{cf}\,\langle {}^\lambda\lambda, \leq^* \rangle$. Cummings and Shelah [7] have generalized Hechler's result as follows:

Theorem II.10. *(Cummings-Shelah) Let λ be a regular cardinal satisfying $\lambda^{<\lambda} = \lambda$, and suppose that $\mathbb{Q}$ is any well-founded poset in which $\mathfrak{b}(\mathbb{Q}) \geq \lambda^+$. Then there is a forcing $\mathbb{D}(\lambda, \mathbb{Q})$ satisfying the following:*

1) $\mathbb{D}(\lambda, \mathbb{Q})$ is λ-closed and λ^+-c.c.;

2) $1 \Vdash \check{\mathbb{Q}}$ can be cofinally embedded into $\langle {}^{\check\lambda}\check\lambda, \leq^* \rangle$;

3) If $\mathfrak{b}(\mathbb{Q}) = \beta$, then $1 \Vdash \mathfrak{b}\langle {}^{\check\lambda}\check\lambda, \leq^* \rangle = \check\beta$;

4) If $\mathfrak{d}(\mathbb{Q}) = \delta$, then $1 \Vdash \mathfrak{d}\langle {}^{\check\lambda}\check\lambda, \leq^* \rangle = \check\delta$.

By λ-closed, we mean that any decreasing chain of length $< \lambda$ has a lower bound. Since the forcing is both λ-closed and λ^+-c.c., it preserves all cofinalities. Cummings and Shelah go on to show that if we assume GCH, then for any class function F that maps each regular cardinal λ to a triple of cardinals $\langle\beta(\lambda),\delta(\lambda),\mu(\lambda)\rangle$ satisfying

$$\lambda^+ \leq \operatorname{cf}\beta(\lambda) = \beta(\lambda) \leq \operatorname{cf}\delta(\lambda) \leq \delta(\lambda) \leq \mu(\lambda)$$

and

$$\lambda < \operatorname{cf}\mu(\lambda)$$

for all λ, there exists a forcing $\mathbb{P}$, preserving all cardinals and cofinalities, such that in the generic extension, $\mathfrak{b}(\lambda) = \beta(\lambda)$, $\mathfrak{d}(\lambda) = \delta(\lambda)$, and $2^\lambda = \mu(\lambda)$ for all regular λ. By what we will observe in Section 3.1, it follows that if the functions satisfy $(\forall\lambda < \kappa)\,\beta(\lambda) = \delta(\lambda)$ but $\beta(\kappa) < \delta(\kappa)$, then κ cannot be measurable in the generic extension.

2.4 Some Posets in Descriptive Set Theory

Recall that $\mathcal{B}_\alpha(\omega,\leq)$ is the poset of Baire class α functions from ${}^\omega\omega$ to ω ordered pointwise by $\leq$, and $\mathcal{B}_\alpha({}^\omega\omega,\leq^*)$ is the poset of Baire class α functions from ${}^\omega\omega$ to ${}^\omega\omega$ ordered pointwise by $\leq^*$. We will eventually compute the cofinalities of these posets. As we stated earlier, the choice of ${}^\omega\omega$ as the domain for the functions is out of convenience and is not essential.

In the literature, the question of what well-orderings (and more generally, linear orderings) embed into posets similar to $\mathcal{B}_\alpha(\omega,\leq)$ has been investigated. In [12], Elekes and Kunen show that for any Polish space X, a well-ordered sequence of length ξ can be embedded into the poset of continuous functions from X to $\mathbb{R}$ (ordered pointwise) iff $\xi < \omega_1$. In fact, they show that for any *metric* space X, a well-ordered sequence of length ξ can be embedded into the poset iff $\xi < d(X)^+$, where $d(X)$ is the smallest size of a dense subset of X. They then show that the separable metric space

$X = \mathcal{P}(\omega)$ is such that for each $\xi < \omega_2$, there is a well-ordered chain of Baire class 1 functions from X to $\mathbb{R}$ of length ξ. The question of whether there exists a separable metric space in which there are such chains of length ω_2 or longer is independent of ZFC (even assuming $\neg$CH).

In [34] (24.III, Theorem 2′), Kuratowski shows that for any Polish space X, a well-ordered sequence of length ξ can be embedded into the poset of Baire class 1 functions from X to $\mathbb{R}$ iff $\xi < \omega_1$. The same question but with Baire class α functions for any fixed $\alpha \in [2, \omega_1)$ is independent of ZFC [31]. Recently, a characterization has been found [13] of what *linear* orderings can be embedded into the poset of Baire class 1 functions from X to $\mathbb{R}$.

Our original motivation for studying $\mathcal{B}_{\omega_1}({}^\omega\omega, \leq^*)$ was to get insight into the poset used in the definition of *Borel boundedness*. This notion appears in the theory of Borel equivalence relations $E \subseteq {}^\omega\omega \times {}^\omega\omega$ all of whose equivalence classes are countable (which hereafter we call *countable Borel equivalence relations*). Many notions of equivalence in mathematics fit into this framework. An important example is Turing equivalence. Given two such equivalence relations E and F on Polish spaces X and Y respectively, a *Borel reduction* from E to F is a Borel function $f : X \to Y$ satisfying

$$(\forall x_1, x_2 \in {}^\omega\omega)\, x_1 E x_2 \Leftrightarrow f(x_1) F f(x_2).$$

An equivalence relation E is *Borel bounded* [3] iff for each Borel $\varphi : {}^\omega\omega \to {}^\omega\omega$, there exists a Borel $\psi : {}^\omega\omega \to {}^\omega\omega$ which pointwise eventually dominates φ and is $=^*$ constant on E classes. Hence, this is a statement about the relationship between E and $\mathcal{B}_{\omega_1}({}^\omega\omega, \leq^*)$.

A sufficient understanding of which equivalence relations are Borel bounded will solve the long-standing but still open *Union Problem*, which conjectures that the increasing union E of a sequence of hyperfinite countable Borel equivalence relations is hyperfinite. Indeed, such an E is hyperfinite iff it is Borel bounded. By hyperfinite, we mean the increasing union of Borel equivalence relations all of whose equivalence classes are finite. It is currently unknown (in ZFC) whether *any* Borel equivalence relation, all of whose classes are countable, is not Borel bounded. However, Martin's conjecture (a deep problem in computability theory concerning the structure of the Turing degrees) implies that Turing equivalence is not Borel bounded [45]. This is a mysterious situation, because it suggests a connection between two difficult open problems in seemingly unrelated areas.

To investigate $\mathcal{B}_{\omega_1}({}^\omega\omega, \leq^*)$ we must ask precise questions, the most natural being "what is its cofinality?". We will prove Theorem VII.28, which implies the answer is 2^ω. The proof will have a computability theoretic nature. This reinforces the hope that there is a connection between Borel boundedness and computability theory.

Finally, we hope that our techniques can be generalized enough to have implications for the *hierarchy of norms* (also called the Steel hierarchy) [35]. This is the poset of surjections $\varphi : {}^\omega\omega \to \alpha$ to ordinals ordered by $\varphi \leq_{\text{FPT}} \psi$ iff there exists a continuous $f : {}^\omega\omega \to {}^\omega\omega$ satisfying

$$(\forall x \in {}^\omega\omega)\, \varphi(x) \leq_{\text{FPT}} \psi(f(x)).$$

The FPT stands for "First Periodicity Theorem". This poset is important when one assumes the axiom of determinacy. If the encoding theorems in this thesis could be sufficiently generalized, we would have (assuming AD) that for each limit ordinal $\alpha < \Theta$ and for each $A \subseteq \omega$, there is some $\varphi_A : {}^\omega\omega \to \alpha$ such that whenever $\psi : {}^\omega\omega \to \alpha$ satisfies $\varphi \leq_{\text{FPT}} \psi$, then $A \in L[c]$ where c is any "code" for ψ.

2.5 Real-valued Measurable Cardinals

Recall that a cardinal δ is *real-valued measurable* if there is a real-valued function $\mu : \mathcal{P}(\delta) \to \mathbb{R}$ satisfying the following:

1) $\mu(\delta) = 1$;

2) $(\forall x \in \delta)\, \mu(\{x\}) = 0$;

3) $(\forall \lambda < \delta)$ if $\langle A_\alpha : \alpha < \lambda \rangle$ is a sequence of pairwise disjoint subsets of δ, then

$$\mu(\bigcup_{\alpha<\lambda} A_\alpha) = \sum_{\alpha<\lambda} \mu(A_\alpha).$$

Given a real-valued measurable cardinal δ, the following are equivalent:

1) δ is not measurable;

2) $\delta \leq 2^\omega$;

3) There exists a function μ witnessing that δ is real-valued measurable such that if $A \subseteq \delta$ satisfies $\mu(A) > 0$, then there exists some $B \subseteq A$ such that $\mu(B) > 0$ and $\mu(A - B) > 0$.

When 2^ω is a real-valued measurable cardinal, we can compute the cofinality of $\langle {}^\lambda\kappa, \leq \rangle$ whenever $\kappa \leq \lambda \leq 2^\omega$ and $\kappa \neq 2^\omega$. Especially notable is that $\mathrm{cf}\, \langle {}^\lambda\kappa, \leq \rangle < 2^\lambda$ when κ is a regular uncountable cardinal $< 2^\omega$ and $\lambda \in [\kappa, 2^\omega)$. We will summarize these known facts now.

Fact II.11. *If 2^ω is real-valued measurable and $\kappa < 2^\omega$, then $2^\kappa = 2^\omega$.*

This is due to Prikry [40]. A proof can be found in Fremlin's article on real-valued measurable cardinals [17]. When we discuss independent families of functions, we will see that $\lambda^\kappa = \lambda$ implies $\mathrm{cf}\, \langle {}^\lambda\kappa, \leq \rangle = 2^\lambda$. Hence, if $\lambda = 2^\omega$ is real-valued measurable and $\kappa < 2^\omega$, then $\mathrm{cf}\, \langle {}^\lambda\kappa, \leq \rangle = 2^\lambda$.

Fact II.12. *If* 2^ω *is real-valued measurable, then* $\mathrm{cf}\,\langle {}^\omega\omega, \leq \rangle < 2^\omega$*.*

This is due to Kunen [32]. In [43], Szymański shows the stronger result that if there exists a σ-additive probability measure on $\mathcal{P}(2^\omega)$ such that each measure 1 set has size 2^ω, then $\mathrm{cf}\,\langle {}^\omega\omega, \leq \rangle < 2^\omega$.

Fact II.13. *If* 2^ω *is real-valued measurable and* $\omega < \lambda < 2^\omega$*, then* $\mathrm{cf}\,\langle {}^\lambda\omega, \leq \rangle = 2^\omega$*.*

The case where $\lambda = \omega_1$ is due to Jech and Prikry [27]. The general case is proved in [43]. In [43], the unnecessary requirement is made that λ be regular.

Fact II.14. *If* 2^ω *is real-valued measurable and* $\omega < \kappa \leq \lambda < 2^\omega$ *with* κ *regular, then* $\mathrm{cf}\,\langle {}^\lambda\kappa, \leq \rangle < 2^\omega$*.*

This is proved in [43].

2.6 Almost Disjoint Functions

Although the question of whether $\mathrm{cf}\,\langle {}^\lambda\kappa, \leq \rangle = 2^\lambda$ for cardinals $\kappa < \lambda$ has not had much attention in the literature, the related problem of constructing large almost disjoint families of functions has been well studied. First, we will explain the connection between the two problems, which ultimately comes from the Unbounded Subset Bound (Proposition II.3). Then, we will survey some standard ways of creating large almost disjoint families. All the significant results in this section can be found in [27].

Definition II.15. Let λ and κ be infinite cardinals. Let $\mathcal{I}$ be an ideal on λ. A family $\mathcal{F} \subseteq {}^\lambda\kappa$ is $\mathcal{I}$-*disjoint* if for distinct $f_1, f_2 \in \mathcal{F}$,

$$\{x \in \lambda : f_1(x) = f_2(x)\} \in \mathcal{I}.$$

If $\mathcal{I}$ is the ideal of bounded subsets of λ, then we call $\mathcal{F}$ an *almost disjoint* family.

This is why we care about $\mathcal{I}$-disjoint families:

Lemma II.16. *Let $\mathcal{I}$ be a κ^+-complete ideal on λ and let $\mathcal{F} \subseteq {}^\lambda\kappa$ be $\mathcal{I}$-disjoint. Then each size κ subset of $\mathcal{F}$ is unbounded in $\langle {}^\lambda\kappa, \leq \rangle$. Hence, assuming $|\mathcal{F}| > \kappa$,*

$$\operatorname{cf} \langle {}^\lambda\kappa, \leq \rangle \geq |\mathcal{F}|.$$

Proof. By Proposition II.3, it suffices to show the first claim. Let $F \subseteq \mathcal{F}$ be a size κ subset of $\mathcal{F}$. Given distinct $f_1, f_2 \in F$, define

$$X_{f_1,f_2} := \{x \in \lambda : f_1(x) = f_2(x)\}.$$

Since there are only κ such X_{f_1,f_2} and $\mathcal{I}$ is κ^+-complete, there exists some $x \in \lambda$ not in any X_{f_1,f_2}. Fix such an x. The values of $f(x)$ for $f \in F$ are all distinct. Hence,

$$\{f(x) : f \in F\}$$

is unbounded in κ. This implies that no single $g \in {}^\lambda\kappa$ can everywhere dominate each $f \in F$. □

This leads us to define the following interval of cardinals:

Definition II.17. Given infinite cardinals λ and κ,

$$\mathrm{ID}(\lambda, \kappa) := \{|\mathcal{F}| : \mathcal{F} \subseteq {}^\lambda\kappa \text{ is } \mathcal{I}\text{-disjoint for some } \kappa^+\text{-complete ideal } \mathcal{I}\}.$$

By the lemma above,

$$\operatorname{cf} \langle {}^\lambda\kappa, \leq \rangle \geq \sup \mathrm{ID}(\lambda, \kappa)$$

(assuming $\kappa^+ \in \mathrm{ID}(\lambda, \kappa)$). There are various ways to prove that $\sup \mathrm{ID}(\lambda, \kappa)$ is large. We will present some now.

Lemma II.18. *There exists a size λ^+ almost disjoint family $\mathcal{F}$ of functions from λ to λ. Hence,*

$$\lambda^+ \in \mathrm{ID}(\lambda, \lambda).$$

Proof. The constant functions form an almost disjoint family of size λ. By diagonalization, no size λ almost disjoint family can be maximal. □

Lemma II.19. *There exists a size κ^+ almost disjoint family $\mathcal{F}$ of functions from κ^+ to κ. Hence,*

$$\kappa^+ \in \mathrm{ID}(\kappa^+, \kappa).$$

Proof. Using the Axiom of Choice, we may easily construct $\mathcal{F} = \langle f_\alpha : \alpha < \kappa^+ \rangle$ such that for each $\alpha < \kappa^+$, the values of $f_\beta(\alpha)$ for $\beta < \alpha$ are distinct from one another. □

Lemma II.20. *There exists a size 2^λ almost disjoint family $\mathcal{F}$ of functions from λ to $2^{<\lambda}$. Hence,*

$$2^\lambda \in \mathrm{ID}(\lambda, 2^{<\lambda}),$$

and therefore $\max \mathrm{ID}(\lambda, 2^{<\lambda}) = 2^\lambda$.

Proof. There are 2^λ paths through the tree ${}^{<\lambda}2$. By injecting each level into $2^{<\lambda}$, we may easily create the desired family. □

These last three propositions are basic building blocks for constructing $\mathcal{I}$-disjoint families of functions. There are also methods for creating new families from old ones, which we will present now.

Lemma II.21 (Tensor Lemma)**.** *If $\mu \in \mathrm{ID}(\lambda, \kappa)$ and $\nu \in \mathrm{ID}(\lambda, \mu)$, then $\nu \in \mathrm{ID}(\lambda, \kappa)$.*

Proof. Let

$$\mathcal{F}_1 = \{f_{1,\alpha} \in {}^\lambda\kappa : \alpha < \mu\}$$

and $\mathcal{I}_1 \subseteq \mathcal{P}(\lambda)$ witness that $\mu \in \mathrm{ID}(\lambda, \kappa)$. Let

$$\mathcal{F}_2 = \{f_{2,\beta} \in {}^\lambda\mu : \beta < \nu\}$$

and $\mathcal{I}_2 \subseteq \mathcal{P}(\lambda)$ witness that $\nu \in \mathrm{ID}(\lambda, \kappa)$.

Let $\mathcal{I}_2 \otimes \mathcal{I}_1$ be the κ-complete ideal on $\lambda \times \lambda$ defined by

$$X \in \mathcal{I}_2 \otimes \mathcal{I}_1 \Leftrightarrow ((\mathcal{I}_2 \otimes \mathcal{I}_1)^* \langle x_2, x_1 \rangle)\, \langle x_2, x_1 \rangle \notin X :\Leftrightarrow (\mathcal{I}_2^* x_2)(\mathcal{I}_1^* x_1)\, \langle x_2, x_1 \rangle \notin X.$$

By $\mathcal{I}^*$, we mean the filter dual to $\mathcal{I}$. By $(\mathcal{S}x)\, \phi(x)$ we mean $\{x : \phi(x)\} \in \mathcal{S}$. For each $\beta < \nu$, let $f_\beta : \lambda \times \lambda \to \kappa$ be the function

$$f_\beta(x_1, x_2) := f_{1, f_{2,\beta}(x_2)}(x_1).$$

Now, for distinct $\beta_1, \beta_2 < \nu$,

$$\begin{aligned} & (\mathcal{I}_2^* x_2)\, f_{2,\beta_1}(x_2) \neq f_{2,\beta_2}(x_2) \\ \Rightarrow\ & (\mathcal{I}_2^* x_2)\, f_{1, f_{2,\beta_1}(x_2)} \text{ is } \mathcal{I}_1\text{-disjoint from } f_{1, f_{2,\beta_2}(x_2)} \\ \Rightarrow\ & (\mathcal{I}_2^* x_2)(\mathcal{I}_1^* x_1)\, f_{1, f_{2,\beta_1}(x_2)}(x_1) \neq f_{1, f_{2,\beta_2}(x_2)}(x_1) \\ \Rightarrow\ & f_{\beta_1} \text{ is } \mathcal{I}_2 \otimes \mathcal{I}_1\text{-disjoint from} f_{\beta_2}. \end{aligned}$$

Thus, $\{f_\beta \in {}^{\lambda \times \lambda}\kappa : \beta < \nu\}$ is an $\mathcal{I}_1 \otimes \mathcal{I}_2$-disjoint family of functions. By bijecting λ with $\lambda \times \lambda$, we get the desired family of functions from λ to κ, and so $\nu \in \mathrm{ID}(\lambda, \kappa)$. □

Lemma II.22 (Crusher Lemma 1)**.** *If $\nu \in \mathrm{ID}(\lambda, \kappa)$, $\lambda < \mathrm{cf}(\kappa)$, and $\mathrm{cf}(\kappa) < \mathrm{cf}(\nu)$, then $(\exists \alpha < \kappa)\, \nu \in \mathrm{ID}(\lambda, \alpha)$. Moreover, if $\mathcal{F} \subseteq {}^\lambda\kappa$ witnesses that $\nu \in \mathrm{ID}(\lambda, \kappa)$, then there exists $\alpha < \kappa$ and a size ν subfamily $\mathcal{G} \subseteq \mathcal{F}$ satisfying $\mathcal{G} \subseteq {}^\lambda\alpha$.*

Proof. This is easy. □

The following hypothesis is needed for the second crusher lemma.

Definition II.23. A family $\mathcal{F} \subseteq {}^\lambda\kappa$ is *branching* if it is almost disjoint and moreover whenever $f_1, f_2 \in \mathcal{F}$ and $\alpha < \lambda$ satisfies $f_1(\alpha) \neq f_2(\alpha)$, then $(\forall \beta > \alpha)\, f_1(\beta) \neq f_2(\beta)$.

Equivalently, $\mathcal{F} \subseteq {}^\lambda\kappa$ is a branching family iff it is included in the set of paths through some tree $T \subseteq {}^{<\lambda}\kappa$ all of whose levels have size $\leq \kappa$. The families given by Lemma II.19 and Lemma II.20 can be assumed to be branching.

Lemma II.24 (Crusher Lemma 2)**.** *Let $\mathcal{F} \subseteq {}^\lambda\kappa$ be branching of size ν. Suppose $\mathrm{cf}(\kappa) < \mathrm{cf}(\lambda)$, $\mathrm{cf}(\kappa) < \mathrm{cf}(\nu)$, and $\lambda < \kappa$. Then there is some size ν subfamily $\mathcal{G} \subseteq \mathcal{F}$ satisfying*

$$(\forall \beta < \kappa)\,|\{f(\beta) : f \in \mathcal{G}\}| \leq \alpha$$

for some $\alpha < \kappa$. Hence, there is a size ν branching subfamily of ${}^\lambda\alpha$.

Proof. Let $\langle \alpha_\gamma : \gamma < \mathrm{cf}(\kappa)\rangle$ be cofinal in κ. For each $f \in \mathcal{F}$, let $\gamma_f < \mathrm{cf}(\kappa)$ satisfy $f(\beta) < \alpha_{\gamma_f}$ for λ many $\beta < \lambda$. Since $\mathrm{cf}(\kappa) < \mathrm{cf}(\lambda)$, these γ_f do in fact exist. Since $\mathrm{cf}(\kappa) < \mathrm{cf}(\nu)$, there is some size ν family $\mathcal{G} \subseteq \mathcal{F}$ and some $\gamma < \mathrm{cf}(\kappa)$ such that $(\forall f \in \mathcal{G})\, f(\beta) < \alpha_\gamma$ for λ many $\beta < \lambda$. We claim that $(\forall \beta < \kappa)\,|\{f(\beta) : f \in \mathcal{G}\}| \leq \lambda \cdot \alpha_\gamma$.

Pick any β. For each $\eta \in \{f(\beta) : f \in \mathcal{G}\}$, let $\langle x_\eta, y_\eta\rangle$ be such that $x_\eta > \beta$ and there exists some $f \in \mathcal{G}$ satisfying $f(\beta) = \eta$ and $f(x_\eta) = y_\eta < \alpha_\gamma$. The pair $\langle x_\eta, y_\eta\rangle$ is well-defined because $(\forall f \in \mathcal{G})\, f(x) < \alpha_\gamma$ for λ many x. Now, the function $\eta \mapsto \langle x_\eta, y_\eta\rangle$ must be an injection (because $\mathcal{G}$ is a branching family). Hence, $|\{f(\beta) : f \in \mathcal{G}\}| \leq \lambda \cdot \alpha_\gamma$. □

We will now give an example of how to apply these lemmas. Let λ be a cardinal and assume $2^{<\lambda} < \aleph_{\mathrm{cf}(\lambda)}$ and $2^{<\lambda} < \mathrm{cf}(2^\lambda)$. Applying Lemma II.20, we get a size 2^λ branching subfamily of functions from λ to $2^{<\lambda}$. Note that each cardinal $< \aleph_{\mathrm{cf}(\lambda)}$ is either regular or has cofinality $< \mathrm{cf}(\lambda)$. This allows us to apply the Crusher Lemmas repeatedly until we get a size 2^λ branching family $\mathcal{G}$ of functions from λ to λ. If in particular $\lambda = \omega_1$, then at the end we may apply the Tensor Lemma with $\mathcal{G}$ and a size ω_1 almost disjoint family of functions from ω_1 to ω to conclude that $\max \mathrm{ID}(\omega_1, \omega) = 2^{\omega_1}$. Hence, $\mathrm{cf}\,\langle {}^{\omega_1}\omega, \leq\rangle = 2^{\omega_1}$.

In [27] (as well as [26]), it is shown how to replace the hypothesis $2^{<\omega_1} < \mathrm{cf}(2^{\omega_1})$ with the weaker one that $2^{<\omega_1} < 2^{\omega_1}$. Let us summarize that $\mathrm{cf}\,\langle {}^{\omega_1}\omega, \leq\rangle = 2^{\omega_1}$

whenever either of the following hold:

1) $2^\omega \leq \omega_2$;

2) $2^\omega < 2^{\omega_1}$ and $2^\omega < \aleph_{\omega_1}$.

Also, given cardinal arithmetic assumptions, it is shown in [27] that there exist large almost disjoint families when there do not exist inner models with large cardinals (by applying a covering theorem).

2.7 Independent Families of Functions

To show $\mathrm{cf}\,\langle {}^\lambda\kappa, \leq\rangle = 2^\lambda$, by the unbounded subset bound (Proposition II.3) it suffices to construct a size 2^λ family $\mathcal{F} \subseteq {}^\lambda\kappa$ all of whose size κ subsets are unbounded in $\langle {}^\lambda\kappa, \leq\rangle$. There are two main ways to get such an $\mathcal{F}$:

1) $\mathcal{F}$ can be $\mathcal{I}$-almost disjoint for some κ-complete ideal on λ;

2) $\mathcal{F}$ can be κ^+-independent.

We will recall the classical theorem which constructs κ^+-independent families. This will give us that $\lambda^\kappa = \lambda$ implies $\mathrm{cf}\,\langle {}^\lambda\kappa, \leq\rangle = 2^\lambda$.

Definition II.25. Let λ, κ, and ν be infinite cardinals. A family $\mathcal{F} \subseteq {}^\lambda\kappa$ is said to be ν-independent if

$$(\forall F \in [\mathcal{F}]^{<\nu})(\forall \varphi : F \to \kappa)(\exists x \in \lambda)(\forall f \in F)\, f(x) = \varphi(f).$$

That is, a family $\mathcal{F} \subseteq {}^\lambda\kappa$ is ν-independent if the functions in each size $< \nu$ subset take specified values at some point $x \in \lambda$. Another name for this is "a family with ν-oscillations" [5]. From the definition, it is clear that if $\mathcal{F} \subseteq {}^\lambda\kappa$ is κ^+-independent, then every size κ subset of $\mathcal{F}$ is unbounded in $\langle {}^\lambda\kappa, \leq\rangle$.

We will now recall an old result to construct such families. For the sake of this section, let $I(\lambda, \kappa, \nu, \mu)$ be the statement "there exists a family $\mathcal{F} \subseteq {}^{\lambda}\kappa$ that is ν-independent and of size μ". $I(\omega, 2, \omega, 2^{\omega})$ and $I(2^{\omega}, 2, \omega, 2^{2^{\omega}})$ were both shown in [16] by Fichtenholz and Kantorovitch. For an arbitrary infinite λ, $I(\lambda, 2, \omega, 2^{\lambda})$ was shown in [20] by Hausdorff. For infinite cardinals λ and κ such that $2^{<\kappa} \leq \lambda$, $I(\lambda, 2, \kappa, 2^{\lambda})$ was shown in [44] by Tarski. Finally, for infinite cardinals λ and κ such that $\lambda^{<\kappa} = \lambda$, $I(\lambda, \lambda, \kappa, 2^{\lambda})$ was shown in [14] by Engelking and Kartowicz. We state this last result as the theorem below. For a proof of this theorem, see (a) $\Rightarrow$ (d) of Theorem 3.16 in [5]. In the next chapter, we will present an instance of this proof in order to analyze the complexity of the functions involved. See also the end of Chapter 3 in [5] for more information.

Theorem II.26. *If $\lambda^{\kappa} = \lambda$, then there is a κ^{+}-independent family of 2^{λ} functions from λ to κ. More generally, if $\lambda^{<\kappa} = \lambda$, then there is a κ-independent family of 2^{λ} functions from λ to κ.*

Note that the following statements are equivalent (for $\kappa \leq \lambda$):

1) $\lambda^{\kappa} = \lambda$;

2) $I(\lambda, \lambda, \kappa^{+}, 2^{\lambda})$;

3) $I(\lambda, \lambda, \kappa^{+}, \kappa)$.

That is, the theorem gives that 1) implies 2). We see that 2) trivially implies 3). Finally, 3) implies 1) because given an $F \subseteq {}^{\lambda}\lambda$ that is κ^{+}-independent of size κ, every $\varphi : F \to \kappa$ corresponds to a unique $x \in \lambda$. Here is the corollary of the theorem relevant to us:

Corollary II.27. *If $\lambda^{\kappa} = \lambda$, then* $\operatorname{cf} \langle {}^{\lambda}\kappa, \leq \rangle = 2^{\lambda}$.

This corollary was surely known by anyone aware of theorem, but the author could find no reference for it. With the special case $\lambda = 2^\omega$ and $\kappa = \omega$, we have the following:

Corollary II.28. *The cofinality of the set of all functions from* 2^ω *to* ω *ordered by everywhere domination is* 2^{2^ω}. *That is,* $\operatorname{cf}\operatorname{All}(\omega, \leq) = 2^{2^\omega}$.

This then has a simple corollary:

Corollary II.29. *Assume* CH. *Then* $\operatorname{cf}\langle {}^{\omega_1}\omega, \leq\rangle = 2^{\omega_1}$.

This is attributed to Kunen (as stated in [27]). Note that this corollary is implied by the comments at the end of the previous section. Hence, there are two quite different proofs that $\operatorname{cf}\langle {}^{\omega_1}\omega, \leq\rangle = 2^{\omega_1}$ assuming CH. From Corollary V.21, we will see a third completely different proof of this.

The existence of sufficiently independent families of functions has an implication for the theory of challenge-response relations. Recall that given $\mathcal{R} = \langle R_-, R_+, R\rangle$, the cardinal $||\mathcal{R}^\perp||$ is the smallest size of a set of challenges $X \subseteq R_-$ not met by a single response $y \in R_+$.

Proposition II.30. *Let* $\mathcal{R} = \langle R_-, R_+, R\rangle$ *be a challenge response relation. Let* $\kappa = ||\mathcal{R}^\perp||$. *Let* λ *be a cardinal satisfying* $\lambda^\kappa = \lambda$. *Let* $\tilde{\mathcal{R}} := \langle {}^\lambda R_-, {}^\lambda R_+, \tilde{R}\rangle$ *be the conjunction of* $\mathcal{R}$ *with itself* λ *many times. That is,* $f\tilde{R}g$ *iff* $(\forall x \in \lambda)\, f(x)Rg(x)$. *Then* $||\tilde{\mathcal{R}}|| = 2^\lambda$. *In fact, there is a set* $\mathcal{F} \subseteq {}^\lambda R_-$ *of size* 2^λ *such that for each size* κ *subset* $\mathcal{F}'$ *of* $\mathcal{F}$, *there is no* $g \in {}^\lambda R_+$ *meeting each element of* $\mathcal{F}'$.

Proof. Let $A = \{a_\alpha : \alpha < \kappa\} \subseteq R_-$ be a set of κ challenges not met by any single response $b \in R_+$. Using Theorem II.26, we obtain a set $\mathcal{F} = \{f_\beta : \beta < 2^\lambda\} \subseteq {}^\lambda R_-$ of size 2^λ such that for each injection $i : \kappa \to 2^\lambda$, there exists an $x \in \lambda$ satisfying

$$(\forall \alpha < \kappa)\, f_{i(\alpha)}(x) = a_\alpha.$$

The set $\mathcal{F}$ is as desired. □

2.8 Dominating Tree Branches

There is an important situation involving trees where the domination relation is relevant. Specifically, let λ and κ be infinite cardinals and $T \subseteq {}^{<\lambda}\kappa$ be a tree. Suppose $f \in {}^{\lambda}\kappa$ is in $[T]$. If $g \in {}^{\lambda}\kappa$ everywhere dominates f, then f is also a path through the tree

$$T_{\leq g} := \{t \in T : (\forall \alpha \in \mathrm{Dom}(t))\, t(\alpha) \leq g(\alpha)\}.$$

Thus, to certify that $[T] \neq \emptyset$, it suffices to find a function $g \in {}^{\lambda}\kappa$ satisfying $[T_{\leq g}] \neq \emptyset$. This is interesting, because it breaks the problem of certifying that $[T] \neq \emptyset$ into two steps:

1) Find a function $g \in {}^{\lambda}\kappa$ sufficiently high up in the ordering $\langle {}^{\lambda}\kappa, \leq \rangle$.

2) Certify that $[T_{\leq g}] \neq \emptyset$.

Recall that a set $A \subseteq \omega$ is Π^1_1 iff there exists a computable function $F : \omega \to \mathcal{P}({}^{<\omega}\omega)$ such that each $F(n)$ is a tree and

$$n \in A \Leftrightarrow [F(n)] = \emptyset.$$

By computable, we mean the set $\{(n,t) : t \in F(n)\} \subseteq \omega \times {}^{<\omega}\omega$ is computable. Fix such an A and F. By hanging each tree $F(n)$ below a stem of length n, we may assume that each $F(n)$ has a stem consisting of 0's of length at least n. Now, for each n such that $[F(n)] \neq \emptyset$, choose some $p_n \in [F(n)]$. Let $g \in {}^{\omega}\omega$ everywhere dominate each p_n (which is possible by the assumption on the $F(n)$'s). The statement

$[F(n)_{\leq g}] = \emptyset$ is Σ^0_1 as a relation of n and g. That is, by compactness, $[F(n)_{\leq g}] = \emptyset$ iff

$$(\exists l \in \omega)(\forall t \in {}^l\omega)\, t \notin F(n)_{\leq g}.$$

It is not difficult to see (using the same trick) that in fact g can be chosen to be Π^1_1 in A. We have just proved the following:

Proposition II.31. *Suppose $A \subseteq \omega$ is Π^1_1. Then there is some $g \in \Pi^1_1 \cap {}^\omega\omega$ such that for any $g' \geq g$, A is (uniformly) Σ^0_1 in g'.*

Hence, we get the existence of the following morphism:

$$\begin{array}{ccc} \Pi^1_1 \cap {}^\omega\omega & \leq & \Pi^1_1 \cap {}^\omega\omega \\ \uparrow & \Downarrow & \downarrow \\ \Pi^1_1 \cap \mathcal{P}(\omega) & \leq_{\Sigma^0_1} & \Pi^1_1 \cap \mathcal{P}(\omega). \end{array}$$

Of course, making finite modifications to g' does not change which sets are $\leq_{\Sigma^0_1}$ below it, so we can replace the top relation $\leq$ with $\leq^*$, but never mind this. This morphism be viewed as an encoding theorem: a Π^1_1 set can be encoded into a function from ω to ω, and that set can be guessed from any dominator of that function (by guessing all sets Σ^0_1 in the dominator). Our encoding theorems have this same spirit, although the proofs are completely different.

Now $\langle {}^\omega\omega, \leq \rangle$ is directed, a set is Δ^1_1 (also called hyperarithmetical) iff both it and its complement are Π^1_1, and $\leq_{\Delta^0_1}$ is the same as Turing reduction $\leq_T$. Thus, we get the following:

Corollary II.32. *Suppose $A \subseteq \omega$ is Δ^1_1. Then there is some $g \in \Delta^1_1 \cap {}^\omega\omega$ such that for any $g' \geq g$, A is (uniformly) computable from g'. Hence,*

$$\begin{array}{ccc} \Delta^1_1 \cap {}^\omega\omega & \leq & \Delta^1_1 \cap {}^\omega\omega \\ \uparrow & \Downarrow & \downarrow \\ \Delta^1_1 \cap \mathcal{P}(\omega) & \leq_T & \Delta^1_1 \cap \mathcal{P}(\omega). \end{array}$$

This result is due to Jockush [28] and Solovay [42]. It is optimal in the sense that for each $A \subseteq \omega$ that is not Δ^1_1 and each $g \in {}^\omega\omega$, there is some $g' \geq g$ that does not compute A. It can be said that the subsets of ω *needed* for $\langle {}^\omega\omega, \leq \rangle$ are precisely those that are Δ^1_1 [1].

The trick we described in this section applies not only to $\langle {}^\omega\omega, \leq \rangle$ but to $\langle {}^\lambda\lambda, \leq \rangle$ whenever λ is strongly inaccessible and has the tree property (a.k.a. weakly compact). In the next chapter, we will describe a slightly different trick where we fix an enumeration of each level of a tree. There, only the tree property and not full weak compactness is what matters.

2.9 Weak Distributivity Laws and Suslin Algebras

The study of properties of complete Boolean algebras is a central area in set theory. From our perspective, it is essentially the same as the theory of forcing. That is, which statements hold in the extension after forcing with a c.B.a. is a property of the c.B.a. and the ground model. Thus, we want to know the effect that axioms (statements in the ground model) have on properties of c.B.a.'s.

Given a challenge-response relation $\mathcal{R} = \langle R_-, R_+, R \rangle$, we may ask which complete Boolean algebras (hereafter called c.B.a.'s) $\mathbb{B}$ are those that after forcing with them, every challenge in the extension is met by a response in the ground model. That is,

$$1 \Vdash_{\mathbb{B}} (\forall x \in R_-)(\exists y \in R_+ \cap \check{V})\, xRy. \tag{2.2}$$

Of course, this statement only makes sense when the forcing extension has its own version of $\mathcal{R}$. We generally assume the relation is sufficiently absolute (so that it means what we expect in the extension). If $\mathbb{B}$ and $\mathcal{R}$ satisfy (2.2), then let us say $\mathbb{B}$ is $\mathcal{R}$-adequate. Fixing $\mathcal{R}$, this gives us a *property of c.B.a.'s*.

If there is a morphism from one relation $\mathcal{R}_1$ to another $\mathcal{R}_2$, and the morphism is sufficiently absolute, then any $\mathbb{B}$ that is $\mathcal{R}_1$-adequate is also $\mathcal{R}_2$-adequate. Hence, the program to find morphisms between (useful) challenge-response relations is a combinatorial approach to finding relationships between properties of c.B.a.'s.

We mention all this because various results on the combinatorial nature of domination are inherent in discussions of *distributivity laws* for c.B.a.'s. As defined in [26], given infinite cardinals λ and κ, we say that a c.B.a. $\mathbb{B}$ is (λ, κ)-*distributive* if

$$\prod_{\alpha<\lambda} \sum_{\beta<\kappa} u_{\alpha,\beta} = \sum_{f:\lambda\to\kappa} \prod_{\alpha<\lambda} u_{\alpha,f(\alpha)}$$

for any $\langle u_{\alpha,\beta} \in \mathbb{B} : \alpha < \lambda, \beta < \kappa\rangle$. Given maximal antichains $A_1, A_2 \subseteq \mathbb{B}$, we say that A_2 *refines* A_1 if $(\forall a_2 \in A_2)(\exists a_1 \in A_1)\, a_2 \leq_{\mathbb{B}} a_1$. It is a fact that $\mathbb{B}$ is (λ, κ)-distributive iff each size λ collection of size κ maximal antichains in $\mathbb{B}$ has a common refinement. Hence, $\mathbb{B}$ is (λ, κ)-distributive for every cardinal κ iff it is $(\lambda, |\mathbb{B}|)$-distributive. This is also called being (λ, ∞)-distributive. There is an important characterization in terms of forcing (which can be found in [26] as Theorem 15.38), which is why we care about (λ, κ)-distributivity:

Fact II.33. *A complete Boolean algebra $\mathbb{B}$ is (λ, κ)-distributive iff*

$$1 \Vdash_{\mathbb{B}} (\forall f : \check{\lambda} \to \check{\kappa})\, f \in \check{V}.$$

Unfortunately, the definition of *weak distributivity* varies in the literature (for example [29]). We will be using the one given by Jech (see [26]). That is, we say that a c.B.a. $\mathbb{B}$ is *weakly* (λ, κ)-*distributive* if

$$\prod_{\alpha<\lambda} \sum_{\beta<\kappa} u_{\alpha,\beta} = \sum_{g:\lambda\to\kappa} \prod_{\alpha<\lambda} \sum_{\beta<g(\alpha)} u_{\alpha,\beta}$$

for any $\langle u_{\alpha,\beta} \in \mathbb{B} : \alpha < \lambda, \beta < \kappa\rangle$. Of course, this also has a characterization in terms of refining antichains. The following connects everywhere domination to weak distributivity of c.B.a.'s:

Fact II.34. *A complete Boolean algebra $\mathbb{B}$ is weakly (λ, κ)-distributive iff*

$$1 \Vdash_{\mathbb{B}} (\forall f : \check{\lambda} \to \check{\kappa})(\exists g : \check{\lambda} \to \check{\kappa})\, g \in \check{V} \wedge f \leq g.$$

For an introduction to distributive laws in c.B.a.'s, see [25]. There are games related to distributive laws in c.B.a.'s. There are implications between distributive laws and players either having or not having winning strategies for these games. In addition to [25], see [9] for a systematic investigation of these properties. There is a large and still growing body of literature on the subject.

The following is often mentioned when discussing distributivity laws for c.B.a.'s:

Definition II.35. A c.B.a. is a *Suslin algebra* if it is atomless, (ω, ∞)-distributive, and c.c.c.

It is a theorem of ZFC that there exists a Suslin algebra iff there exists a Suslin tree. Furthermore, given a Suslin algebra $\mathbb{B}$, there is a Suslin tree (turned upside down) that completely embeds into $\mathbb{B}$, so $\mathbb{B}$ is not $(\omega_1, 2)$-distributive (see [26]). If a c.B.a. is c.c.c, then it is also weakly (λ, κ)-distributive for every λ and every regular uncountable κ. We will now recall the proof of a stronger statement. Recall that a forcing has the κ-c.c. if every antichain has size $< \kappa$:

Lemma II.36. *If λ and κ are infinite cardinals with κ regular, $\mathbb{P}$ is a forcing with the κ-c.c, $p \in \mathbb{P}$, $\dot{f} \in V^{\mathbb{P}}$, and $p \Vdash \dot{f} : \check{\lambda} \to \check{\kappa}$, then there is some $g : \lambda \to \kappa$ satisfying*

$$p \Vdash \dot{f} \leq \check{g}.$$

Proof. For each $\alpha < \lambda$, consider the set

$$S_\alpha := \{\beta < \kappa : (\exists p' \leq p)\, p' \Vdash \dot{f}(\check{\alpha}) = \check{\beta}\}.$$

Since $\mathbb{P}$ has the κ-c.c., it must be that each S_α has size $< \kappa$. For each $\alpha < \lambda$, define

$$g(\alpha) := \sup S_\alpha.$$

If it was not the case that $p \Vdash (\forall\alpha < \check{\lambda})\, \dot{f}(\alpha) \leq \check{g}(\alpha)$, then there would be some $p' \leq p$, $\alpha < \lambda$, and $\beta < \kappa$ satisfying

$$p' \Vdash \dot{f}(\check{\alpha}) = \check{\beta} > \check{g}(\check{\alpha}),$$

but this would contradict the definition of g. □

Corollary II.37. *If a c.B.a. is κ-c.c, then for each λ it is weakly (λ, κ)-distributive.*

The problem of finding weakly (λ, κ)-distributive c.B.a.'s which are not κ-c.c. is somewhat of a mystery. Now, the lemma above gives us that $p \Vdash \dot{f} \leq \check{g}$ and not just that there exists some $p' \leq p$ satisfying $p' \Vdash \dot{f} \leq \check{g}$. This important point gives us the next corollary. We are not being pedantic: there is consistently, relative to large cardinals, a forcing $\mathbb{P}$ (see [36]) which does not collapse cardinals and does not add reals, but still

$$1 \Vdash \mathrm{cf}\,\langle {}^{\omega}\omega, \leq\rangle < \check{\overbrace{\mathrm{cf}\,\langle {}^{\omega}\omega, \leq\rangle}}.$$

If there exists a forcing with this property, then the following hold: (by [36])

1) the forcing must collapse some cardinal's cofinality;

2) there exists an inner model with a measurable cardinal.

Statement 2) follows from 1) and the fact that the forcing does not collapse cardinals. This next corollary uses the lemma above for both directions. While the author could not find a reference for the following corollary, it is surely folklore knowledge.

Corollary II.38. *If λ and κ are cardinals with κ regular and $\mathbb{P}$ is a forcing with the κ-c.c, then*

$$1 \Vdash \mathrm{cf}\,\langle {}^{\check{\lambda}}\check{\kappa}, \leq\rangle = \check{\overbrace{\mathrm{cf}\,\langle {}^{\lambda}\kappa, \leq\rangle}}.$$

Proof. Let $\nu := \text{cf}\,\langle {}^{\lambda}\kappa, \leq \rangle$. To see why $1 \Vdash \text{cf}\,\langle {}^{\check{\lambda}}\check{\kappa}, \leq \rangle \leq \check{\nu}$, note that the lemma above implies that if $A \subseteq {}^{\lambda}\kappa$ is cofinal in $\langle {}^{\lambda}\kappa, \leq \rangle$, then $1 \Vdash (\check{A}$ is cofinal in $\langle {}^{\check{\lambda}}\check{\kappa}, \leq \rangle)$.

For the more difficult direction, we must show that in the extension there is no cofinal family of size strictly smaller than $\check{\nu}$. Suppose, towards a contradiction, that $1 \nVdash \text{cf}\,\langle {}^{\check{\lambda}}\check{\kappa}, \leq \rangle \geq \check{\nu}$. Then there exists some $p \in \mathbb{P}$ satisfying $p \Vdash \text{cf}\langle {}^{\check{\lambda}}\check{\kappa}, \leq \rangle < \check{\nu}$. Pick p, $\mu < \nu$, and $\dot{\tau} \in V^{\mathbb{P}}$ so that $p \Vdash (\dot{\tau} : \check{\mu} \to {}^{\check{\lambda}}\check{\kappa})$ and

$$p \Vdash (\forall g \in {}^{\check{\lambda}}\check{\kappa})(\exists \gamma < \check{\mu})\, g \leq \dot{\tau}(\gamma).$$

For each $\gamma < \mu$, applying the lemma above to (a name which 1 forces is equivalent to) $\dot{\tau}(\check{\gamma})$ produces a function $g_\gamma \in {}^{\lambda}\kappa$ satisfying

$$p \Vdash \dot{\tau}(\check{\gamma}) \leq \check{g}_\gamma.$$

We claim that $\{g_\gamma : \gamma < \mu\} \subseteq {}^{\lambda}\kappa$ is cofinal in $\langle {}^{\lambda}\kappa, \leq \rangle$. Once this is shown, we will have the contradiction.

Consider any $g \in {}^{\lambda}\kappa$. We will find $\gamma < \mu$ satisfying $g \leq g_\gamma$. We have $p \Vdash (\exists \gamma < \check{\mu})\, \check{g} \leq \dot{\tau}(\gamma)$. Pick $p' \leq p$ and $\gamma < \mu$ satisfying $p' \Vdash \check{g} \leq \dot{\tau}(\check{\gamma})$. We now have

$$p' \Vdash \check{g} \leq \dot{\tau}(\check{\gamma}) \leq \check{g}_\gamma.$$

Since $p' \Vdash \check{g} \leq \check{g}_\gamma$, we have

$$g \leq g_\gamma.$$

The proof is complete. □

An easily verifiable fact that we should mention is that any forcing which collapses the cofinality of a cardinal λ to $\mu < \lambda$ is not weakly (λ, μ)-distributive.

CHAPTER III

Building on Past Work

This chapter is mostly a continuation of the last, with the difference being these results are new. The last section of this chapter, however, is relevant to the goal of computing $\mathcal{B}_\alpha(\omega, \leq)$ and $\mathcal{B}_\alpha({}^\omega\omega, \leq^*)$ for all $\alpha \leq \omega_1$. We begin with the easy observation that just as GCH cannot first fail at a measurable cardinal, neither can the equality $\mathfrak{b}\,\langle {}^\lambda\lambda, \leq^* \rangle = \mathrm{cf}\,\langle {}^\lambda\lambda, \leq^* \rangle$. Next, we describe a trick relating everywhere domination to the existence of paths through trees of a slightly different nature than the one in the previous chapter. This allows us to make observations such as the following: $2^{\omega_1} = \max\{\mathrm{cf}\,\langle {}^{\omega_1}\omega, \leq \rangle, \mu\}$, where μ is the smallest size of a collection of ω_1-trees $T \subseteq {}^{<\omega_1}\omega$ such that every element of ${}^{\omega_1}\omega$ is a path through one of them. Also, forcing (non-trivially) with an Aronszajn tree is not weakly (ω_1, ω)-distributive.

In the next section, we discuss the relationship between $\langle {}^\lambda\kappa_1, \leq \rangle$ and $\langle {}^\lambda\kappa_2, \leq \rangle$ for $\kappa_1 \neq \kappa_2$. This is surprisingly subtle. After that, we prove a result which implies that whenever λ is a singular strong limit cardinal and $\kappa < \lambda$, then $\mathrm{cf}\,\langle {}^\lambda\kappa, \leq \rangle = 2^\lambda$. At the same time, we discuss the relationship between the poset $\langle {}^\lambda\kappa, \leq \rangle$ and those studied in PCF theory.

Finally, we analyze the complexity of the functions created by an instance of the classical theorem to create large independent families of functions. This allows us to conclude that $\operatorname{cf}\mathcal{B}_\alpha(\omega, \leq) = 2^\omega$ for all but very small $\alpha \leq \omega_1$.

3.1 Scales at a Measurable Cardinal

Given any poset $\mathbb{P} = \langle X, \leq \rangle$, $\mathfrak{b}\,\mathbb{P} = \operatorname{cf}\mathbb{P}$ if and only if $\mathbb{P}$ has a scale. This is interesting because it implies the statement $\mathfrak{b}\,\mathbb{P} = \operatorname{cf}\mathbb{P}$ is equivalent to one which uses different quantifiers. Specifically, the statement $\mathfrak{b}\,\mathbb{P} = \operatorname{cf}\mathbb{P}$ appears to involve a universal quantification over all subsets of $\mathbb{P}$. On the other hand, the statement that $\mathbb{P}$ has a scale is asserting the existence of a sequence $\langle x_\alpha \in \mathbb{P} : \alpha < \kappa \rangle$ satisfying

$$[(\forall \alpha < \beta < \kappa)\, x_\alpha \leq x_\beta] \wedge [(\forall x \in X)(\exists \alpha < \kappa)\, x \leq x_\alpha].$$

This is second order existential quantification over $\mathbb{P}$ followed by first order quantification. This implies that having a scale is upwards absolute:

Observation III.1. *Let M be a transitive model of ZFC and $\mathbb{P} \in M$ be a poset. If $(\mathbb{P}$ has a scale$)^M$, then $\mathbb{P}$ has a scale.*

Hence, if M is a transitive model of ZFC and $\mathbb{P} \in M$ is a posetx, then $(\mathfrak{b}\,\mathbb{P} = \operatorname{cf}\mathbb{P})^M$ implies $\mathfrak{b}\,\mathbb{P} = \operatorname{cf}\mathbb{P}$. This allows us to conclude the following, which is very similar to the fact that GCH cannot first fail at a measurable cardinal:

Proposition III.2. *Let U be a normal ultrafilter on a measurable cardinal κ. If*

$$\{\lambda < \kappa : \langle {}^\lambda\lambda, \leq^* \rangle \text{ has a scale}\} \in U,$$

then $\langle {}^\kappa\kappa, \leq^* \rangle$ has a scale.

Proof. Let M be the transitive collapse of the ultrapower of V by U. By Łoś's theorem, $(\langle {}^\kappa\kappa, \leq^* \rangle$ has a scale$)^M$. Since ${}^\kappa M \subseteq M$, we have $\langle {}^\kappa\kappa, \leq^* \rangle^M = \langle {}^\kappa\kappa, \leq^* \rangle$. Combining these two facts with the previous observation, we see that $\langle {}^\kappa\kappa, \leq^* \rangle$ has a scale. □

For each regular cardinal λ, let $\mathfrak{b}(\lambda) := \mathfrak{b}\,\langle {}^{\lambda}\lambda, \leq^* \rangle$ and $\mathfrak{d}(\lambda) := \mathrm{cf}\,\langle {}^{\lambda}\lambda, \leq^* \rangle$. The proposition above shows that if $\{\lambda < \kappa : \mathfrak{b}(\lambda) = \mathfrak{d}(\lambda)\}$ is included in some normal ultrafilter on κ, then $\mathfrak{b}(\kappa) = \mathfrak{d}(\kappa)$.

3.2 More on Dominating Tree Branches

We will present a trick similar to the one in Section 2.8. We hope to convince the reader that the problem of finding paths through trees is significantly related to the everywhere domination relation; trees are an important source of examples to understand $\langle {}^{\lambda}\kappa, \leq \rangle$. Recall the following:

Definition III.3. Let λ be an infinite cardinal and X be a set. A λ-*tree* is a tree all of whose levels have size $< \lambda$.

Definition III.4. A cardinal λ has the *tree property* if every λ-tree T of height λ has a length λ branch.

For notational simplicity, let $\mu \leq \lambda$ be infinite cardinals. Suppose we have a tree $T \subseteq {}^{<\lambda}X$ all of whose levels have size $\leq \mu$, as well as a sequence $\mathcal{S} = \langle \eta_\alpha : \alpha < \lambda \rangle$ such that each η_α is a surjection from μ onto $T \cap {}^{\alpha}X$. Suppose $f \in {}^{\lambda}X$ is in $[T]$. Define the function $f_{\mathcal{S}} \in {}^{\lambda}\mu$ by

$$f_{\mathcal{S}}(\alpha) := \min\{\beta < \mu : \eta_\alpha(\beta) = f \restriction \alpha\}.$$

If $g \in {}^{\lambda}\mu$ everywhere dominates $f_{\mathcal{S}}$, then f is also a path through the tree

$$T^{\leq g} := \{t \in T : (\forall \alpha \in \mathrm{Dom}(t))(\exists \beta \leq g(\alpha))\, \eta_\alpha(\beta) = t \restriction \alpha\}.$$

Note the difference between this definition of $T^{\leq g}$ and the definition of $T_{\leq g}$ in Section 2.8. Of course, $T^{\leq g}$ depends on the sequence $\mathcal{S}$. Like before, we see that to certify that $[T] \neq \emptyset$, it suffices to find a function $g \in {}^{\lambda}\mu$ satisfying $[T^{\leq g}] \neq \emptyset$.

There are two interesting cases. The first is that $\mu = \lambda$ and λ has the tree property. Hence, $T^{\leq g}$ is a λ-tree. This is the situation most analogous to Section 2.8, because $[T^{\leq g}] \neq \emptyset$ iff $T^{\leq g}$ has λ non-empty levels. Hence, $[T] \neq \emptyset$ iff there exists a $g \in {}^{\lambda}\lambda$ such that $T^{\leq g}$ has λ non-empty levels. This shows that testing whether $[T] \neq \emptyset$ breaks into the difficult task of finding a function g sufficiently high up in $\langle {}^{\lambda}\lambda, \leq \rangle$, and the comparatively easy task of testing whether $T^{\leq g}$ has λ non-empty levels.

The other interesting case is that $\mu^+ = \lambda$ (the remaining case that $\mu^+ < \lambda$ trivializes our discussion). Given T and a transitive model M of ZFC with $T \in M$, it cannot be said in general that M contains every element of $[T]$. Indeed, $[T]$ could be non-empty and yet $M \cap [T] = \emptyset$. For example, T could be a Suslin tree in M and V could be a forcing extension of M by T. However, if $g \in {}^{\lambda}\mu \cap M$ and $\mathcal{S} \in M$, then $T^{\leq g} \in M$ and $[T^{\leq g}] \subseteq M$. The second conclusion follows easily from a standard observation:

Lemma III.5. *If $T' \subseteq {}^{<\lambda}\mu$ is a tree with $\mu < \lambda$ both cardinals with λ regular and each level of T' has size $< \mu$, then for each $f \in [T']$ there is some $\alpha < \lambda$ such that f is the only length λ path through T' extending $f \restriction \alpha$.*

Proof. Suppose, towards a contradiction, that there is a set $H \subseteq [T']$ disjoint from $\{f\}$ such that the elements of H deviate from f at levels unbounded in λ. Then since λ is regular and $\mu < \lambda$, we may fix an $\alpha < \lambda$ such that there is a set K of $\geq \mu$ elements of H which deviate from f before level α, and they deviate from f at distinct levels. Then $\{k \restriction \alpha : k \in K\}$ is a set of $\geq \mu$ elements of the α-th level of T', which we assumed had size $< \mu$. □

Corollary III.6. *Let T' be as in the lemma above.*

1) If M is a transitive model of ZFC and $T' \in M$, then $[T'] \subseteq M$.

2) $[T']$ has size at most λ.

The arguments we have given easily show the following:

Proposition III.7. *Let μ be a cardinal, $\lambda := \mu^+$, X be a set, and $T \subseteq {}^{<\lambda}X$ be a tree such that each level of T has size $\leq \mu$. If T, as a forcing, adds a path through T, then it is not weakly (λ, μ)-distributive.*

Proof. Fix an appropriate sequence $\mathcal{S}$ of surjections onto the levels of T. Let f be a path through T added in the forcing extension. If the forcing is weakly (λ, μ)-distributive, then we may fix a $g \geq f_{\mathcal{S}}$ in the ground model. Then the tree $T^{\leq g}$ is in the ground model, f is a path through it, and all paths through $T^{\leq g}$ are in the ground model. □

For example, a pruned Aronszajn tree $T \subseteq {}^{<\omega_1}\omega$ is not weakly (ω_1, ω)-distributive as a forcing.

As a final observation, let $\mathcal{T}$ be a family of minimal cardinality of ω_1-trees such that each $x \in {}^{\omega_1}\omega$ is a path through one of them. Of course, if there are no ω_1-trees with 2^{ω_1} branches, then $|\mathcal{T}| = 2^{\omega_1}$. If there are such trees, then perhaps $|\mathcal{T}| < 2^{\omega_1}$, and in this case we will argue that $\operatorname{cf} \langle {}^{\omega_1}\omega, \leq \rangle = 2^{\omega_1}$. Thus, we claim the following (potentially non-trivial) equality:

$$2^{\omega_1} = \max\{\operatorname{cf} \langle {}^{\omega_1}\omega, \leq \rangle, |\mathcal{T}|\}.$$

Here is the proof: let $\mathcal{G} \subseteq {}^{\omega_1}\omega$ be cofinal in $\langle {}^{\omega_1}\omega, \leq \rangle$ of minimal cardinality. For each $T \in \mathcal{T}$, we have

$$[T] = \bigcup \{[T^{\leq g}] : g \in \mathcal{G}\}.$$

By Lemma III.5, each $[T^{\leq g}]$ has size $\leq \omega_1$, so $[T] \leq |\mathcal{G}|$. Hence, $2^{\omega_1} = \max\{|\mathcal{G}|, |\mathcal{T}|\}$.

3.3 Changing κ

As we noted in the introduction, for a fixed cardinal λ and regular cardinals $\kappa_1 < \kappa_2 \leq \lambda$, there is no immediate reason for there to be any relationship between $\mathrm{cf}\,\langle {}^{\lambda}\kappa_1, \leq\rangle$ and $\mathrm{cf}\,\langle {}^{\lambda}\kappa_2, \leq\rangle$. Indeed, since every size κ_1 set in $\langle {}^{\lambda}\kappa_2, \leq\rangle$ is bounded but this is not the case for $\langle {}^{\lambda}\kappa_1, \leq\rangle$, there cannot exist a morphism from $\langle {}^{\lambda}\kappa_2, \leq\rangle$ to $\langle {}^{\lambda}\kappa_1, \leq\rangle$. However, the following is a way to convert a large number of "κ_1 challenges" into a single "κ_2 challenge". We get an immediate improvement in that we can convert that large number of κ_1 challenges into that same number of κ_2 challenges.

Lemma III.8 (Increasing Range Characterization)**.** *Let λ be an infinite cardinal and let $\kappa_1 < \kappa_2$ be regular cardinals. The following are equivalent:*

1) There exists a size κ_2 family $\mathcal{F} \subseteq {}^{\lambda}\kappa_1$ all of whose size κ_2 subsets are unbounded in $\langle {}^{\lambda}\kappa_1, \leq\rangle$.

2) There exists a morphism from $\langle {}^{\lambda}\kappa_1, \leq\rangle$ to $\langle \kappa_2, \leq\rangle$.

3) There exists a morphism from $\langle {}^{\lambda}\kappa_1, \leq\rangle$ to $\langle {}^{\lambda}\kappa_2, \leq\rangle$.

Proof. First, note that 2) and 3) are equivalent. The 3) implies 2) direction is easiest because there is a morphism from $\langle {}^{\lambda}\kappa_2, \leq\rangle$ to $\langle \kappa_2, \leq\rangle$. For the 2) implies 3) direction, if there was a morphism from $\langle {}^{\lambda}\kappa_1, \leq\rangle$ to $\langle \kappa_2, \leq\rangle$, then there would also be a morphism from $\langle {}^{\lambda\times\lambda}\kappa_1, \leq\rangle$ to $\langle {}^{\lambda}\kappa_2, \leq\rangle$, and of course $\langle {}^{\lambda\times\lambda}\kappa_1, \leq\rangle$ is isomorphic to $\langle {}^{\lambda}\kappa_1, \leq\rangle$.

We will now show that 2) implies 1). Let $\langle \phi_-, \phi_+\rangle$ be a morphism from $\langle {}^{\lambda}\kappa_1, \leq\rangle$ to $\langle \kappa_2, \leq\rangle$. Then $\mathrm{Im}(\phi_-) \subseteq {}^{\lambda}\kappa_1$ has size κ_2, and all its size κ_2 subsets are unbounded in $\langle {}^{\lambda}\kappa_1, \leq \rangle$.

Finally, for the 1) implies 2) direction, fix a size κ_2 family $\{f_\alpha \in {}^{\lambda}\kappa_1 : \alpha < \kappa_2\}$ all of whose size κ_2 subsets are unbounded. We will define the morphism:

$$\begin{array}{ccccc} {}^{\lambda}\kappa_1 & & \leq & & {}^{\lambda}\kappa_1 \\ \phi_- \uparrow & & \Downarrow & & \downarrow \phi_+ \\ \kappa_2 & & \leq & & \kappa_2. \end{array}$$

Define $\phi_-(\alpha) := f_\alpha$ and

$$\phi_+(g) := \sup\{\alpha < \kappa_2 : f_\alpha \leq g\}.$$

Note that ϕ_+ is well-defined by the hypothesis on $\mathcal{F}$. □

The morphisms given by this lemma are destroyed if we force an everywhere dominating function from λ to κ_1, because $\mathcal{F}$ becomes bounded. Indeed, the morphisms are not "canonical". This contrasts with the morphisms we will construct in the main part of this thesis, which are canonical. This next proposition applies the lemma above using two ways of building families $\mathcal{F}$ all of whose size $|\mathcal{F}|$ subsets are unbounded.

Proposition III.9. *Let λ be an infinite cardinal and $\kappa_1 < \kappa_2 \leq \lambda$ be regular cardinals. Assume one of the following:*

1) $\kappa_2^{\kappa_1} \leq \lambda$;

2) $(\exists n \in \omega)\, \kappa_2 = \kappa_1^{\overbrace{+\cdots+}^{n}}$.

Then there exists a morphism from $\langle {}^{\lambda}\kappa_1, \leq \rangle$ to $\langle {}^{\lambda}\kappa_2, \leq \rangle$.

Proof. First assume 1). Let $\mu := \kappa_2^{\kappa_1}$. Since $\mu^{\kappa_1} = \mu$, by Corollary II.27 there is a size 2^μ family $\mathcal{F} \subseteq {}^{\mu}\kappa_1$ that is κ_1^+-independent. Letting $\mathcal{F}' \subseteq \mathcal{F}$ be a size κ_2 subfamily of $\mathcal{F}$, we see that $\mathcal{F}'$ is also κ_1^+-independent. Every size κ_1 subset of $\mathcal{F}'$

is unbounded. Thus, every size κ_2 subset of $\mathcal{F}'$ is unbounded. We may arbitrarily extend the functions in $\mathcal{F}'$ to be defined on all of λ. We now have a size κ_2 family $\mathcal{F}'' \subseteq {}^{\lambda}\kappa_2$ all of whose size κ_2 subsets are unbounded. Applying the lemma above, we are done.

Now assume 2). Since morphisms compose together, we may assume $\kappa_2 = \kappa_1^+$. Let $\mathcal{F} \subseteq {}^{\kappa_2}\kappa_1$ be a size κ_2 family of almost disjoint functions given by Lemma II.19. Note that all size κ_1 (and therefore all size κ_2) subsets of $\mathcal{F}$ are unbounded. Extend each function in $\mathcal{F}$ arbitrarily to obtain a size κ_2 family $\mathcal{F}' \subseteq {}^{\lambda}\kappa_1$ all of whose size κ_2 subsets are unbounded. $\square$

Similarly to 2) in the proposition above, one also gets an appropriate $\mathcal{F}$ provided $\kappa_2 \in \mathrm{ID}(\lambda, \kappa_1)$, where ID is from Definition II.17. We now have a pleasant picture (omitting unnecessary arrows) of some of the morphisms between the first few posets of the form $\langle {}^{\lambda}\kappa, \leq \rangle$:

$$\begin{array}{ccccccccc}
\langle \omega_2, \leq \rangle & \longleftarrow & & & & & \langle {}^{\omega_2}\omega_2, \leq \rangle & \longleftarrow & \cdots \\
 & & & & & & \uparrow & & \\
\langle \omega_1, \leq \rangle & \longleftarrow & & & \langle {}^{\omega_1}\omega_1, \leq \rangle & \longleftarrow & \langle {}^{\omega_2}\omega_1, \leq \rangle & \longleftarrow & \cdots \\
 & & & & \uparrow & & \uparrow & & \\
\langle \omega, \leq \rangle & \longleftarrow & \langle {}^{\omega}\omega, \leq \rangle & \longleftarrow & \langle {}^{\omega_1}\omega, \leq \rangle & \longleftarrow & \langle {}^{\omega_2}\omega, \leq \rangle & \longleftarrow & \cdots
\end{array}$$

We have omitted each $\langle {}^{\lambda}\kappa, \leq \rangle$ where $\lambda < \kappa$ because there are morphisms in both directions between each such a $\langle {}^{\lambda}\kappa, \leq \rangle$ and $\langle \kappa, \leq \rangle$. By the reason we gave at the beginning of this section, there are no arrows from a given row to a strictly lower row. Of course, there are no arrows between $\langle \kappa_1, \leq \rangle$ and $\langle \kappa_2, \leq \rangle$ when $\kappa_1 \neq \kappa_2$ are regular. An example question we may ask is the following: is there an arrow from $\langle {}^{\omega}\omega, \leq \rangle$ to $\langle \kappa, \leq \rangle$ for some regular uncountable κ? This, by Lemma III.8, is equivalent to asking what are the sizes of families $\mathcal{F} \subseteq {}^{\omega}\omega$ all of whose size $|\mathcal{F}|$ subsets are unbounded in $\langle {}^{\omega}\omega, \leq \rangle$. Kunen discusses this in part of [32]. The statement that there is no such family of size κ he calls $D(\kappa)$.

We may ask if there are *any* arrows that go to the right (and also possibly up). Such morphisms would be counterintuitive, but we see no ZFC proof that none exist. Here is an easy argument assuming GCH that none exist: if $\kappa \leq \lambda$ are infinite cardinals, then $\lambda^+ \leq \operatorname{cf} \langle {}^{\lambda}\kappa, \leq \rangle \leq 2^\lambda = \lambda^+$. Thus, given $\lambda_1 < \lambda_2$ with $\kappa_1 \leq \lambda_1$ and $\kappa_2 \leq \lambda_2$ regular, we have $\operatorname{cf} \langle {}^{\lambda_1}\kappa_1, \leq \rangle = \lambda_1^+ < \lambda_2^+ = \operatorname{cf} \langle {}^{\lambda_2}\kappa_2, \leq \rangle$. This prevents there being a morphism from $\operatorname{cf} \langle {}^{\lambda_1}\kappa_1, \leq \rangle$ to $\operatorname{cf} \langle {}^{\lambda_2}\kappa_2, \leq \rangle$.

Finally we must ask if every poset in a lower row has an arrow to a higher row (except in the leftmost column). This appears to be a subtle problem. Each such morphism is an example of "non-reflection" (borrowing the terminology that is used in a significant portion of infinitary combinatorics [6]). The following definition appears to be the relevant concept:

Definition III.10. Let $\mathcal{R} = \langle R_-, R_+, R \rangle$ be a challenge-response relation. Let $\kappa \geq \mu$ be infinite cardinals. We say that $\mathcal{R}$ has (κ, μ)-*non-reflection* if there exists a set of κ challenges such that no μ members are all met by a single response.

We say $\mathcal{R}$ has (κ, μ)-*reflection* just when it does not have (κ, μ)-non-reflection. That is, when for every set of κ challenges, there are μ elements of that set met by a single response. Of course, these definitions are only interesting when there exist μ challenges not met by a single response. If $\mu_1 \leq \mu_2 \leq \kappa$, then $\mathcal{R}$ being (κ, μ_2)-reflecting implies it is (κ, μ_1)-reflecting.

A challenge-response relation having (κ, μ)-non-reflection is the analogue of part 1) of Lemma III.8. By a similar argument to the 1) iff 2) part of that lemma, we see that $\mathcal{R} = \langle R_-, R_+, R \rangle$ has (κ, κ)-non-reflection iff there exists a morphism from $\mathcal{R}$ to $\langle \kappa, \leq \rangle$. Similarly, Given $\mu \leq \kappa$, $\mathcal{R}$ has (κ, μ)-non-reflection iff there exists a morphism from $\mathcal{R}$ to $\langle \kappa, [\kappa]^{<\mu}, \in \rangle$.

3.4 Singular Strong Limit Cardinals

Let λ be a strong limit cardinal and $\kappa < \lambda$ be regular. If λ is regular, then since $\lambda^\kappa = \lambda$, we have $\mathrm{cf}\,\langle {}^\lambda\kappa, \leq \rangle = 2^\lambda$. The question arises whether we can drop the hypothesis that λ be regular. We will first give in Proposition III.11 a direct combinatorial proof that the answer is yes. In fact, the full hypothesis of λ being a strong limit cardinal is not needed. After, we will show that standard PCF theory facts imply most instances of the problem. This is because there exist morphisms from posets of the form $\langle {}^\lambda\kappa, \leq \rangle$ to posets of the form $\langle \prod_{\alpha<\delta} \lambda_\alpha, \leq \rangle$ where $\langle \lambda_\alpha : \alpha < \delta \rangle$ is cofinal in λ and $\mathrm{cf}(\lambda) \leq \delta < \lambda$.

Proposition III.11. *Let λ be a singular cardinal. Let $\kappa < \lambda$ be regular. Assume*

$$(\forall \sigma < \lambda)\, \sigma^\kappa < \lambda$$

and $2^{\mathrm{cf}(\lambda)} < \lambda$. Let $\nu = \max\{(2^\kappa)^+, (2^{\mathrm{cf}(\lambda)})^+\}$. Then there exists a size 2^λ family $\mathcal{F} \subseteq {}^\lambda\kappa$ all of whose size ν subsets are unbounded in $\langle {}^\lambda\kappa, \leq \rangle$. Hence, $\mathrm{cf}\,\langle {}^\lambda\kappa, \leq \rangle = 2^\lambda$.

Proof. Once we construct the family $\mathcal{F}$ so that all its size ν subsets are unbounded, it will follow from the unbounded subset bound (Proposition II.3) that $\mathrm{cf}\,\langle {}^\lambda\kappa, \leq \rangle = 2^\lambda$ (because $\nu < \lambda < 2^\lambda = |\mathcal{F}|$). To begin, let $\langle \lambda_\alpha < \lambda : \alpha < \mathrm{cf}(\lambda) \rangle$ be an increasing sequence with limit λ. Letting

$$T := \prod_{\alpha < \mathrm{cf}(\lambda)} \lambda_\alpha$$

denote the Cartesian product of these sets, we have $|T| = 2^\lambda$. For each $t \in T$ we will define a function f_t, and our final family $\mathcal{F}$ will be

$$\mathcal{F} := \{f_t : t \in T\}.$$

Let $\langle X_\alpha \subseteq \lambda : \alpha < \mathrm{cf}(\lambda)\rangle$ be a sequence of disjoint subsets of λ satisfying $|X_\alpha|^\kappa = |X_\alpha|$ and $\lambda_\alpha \leq 2^{|X_\alpha|}$ for each $\alpha < \mathrm{cf}(\lambda)$. Such a sequence exists because of the assumption that $(\forall \sigma < \lambda)\, \sigma^\kappa < \lambda$. For each $\alpha < \mathrm{cf}(\lambda)$, let F_α be a size λ_α family of functions from X_α to κ all of whose size κ subsets are unbounded in $\langle {}^\lambda\kappa, \leq\rangle$. In fact, since $|X_\alpha|^\kappa = |X_\alpha|$ there is such a family of size $2^{|X_\alpha|}$ (by Corollary II.27).

We are now ready to define our family $\mathcal{F}$. For each $t \in T$, let $f_t \in {}^\lambda\kappa$ be any function such that for each $\alpha < \mathrm{cf}(\lambda)$, $f_t \restriction X_\alpha$ equals the $t(\alpha)$-th element of F_α. Let $\mathcal{F} := \{f_t : t \in T\}$. Of course, $t_1 \neq t_2$ implies $f_{t_1} \neq f_{t_2}$. There is an important way to color pairs from T. Namely, let

$$c : [T]^2 \to \mathrm{cf}(\lambda)$$

be the function which given the pair $\{t_1, t_2\} \in [T]^2$ returns the unique $\alpha = c(\{t_1, t_2\})$ satisfying $t_1(\alpha) \neq t_2(\alpha)$ and $(\forall \beta < \alpha)\, t_1(\beta) = t_2(\beta)$. Given $\{t_1, t_2\} \in [T]^2$ and $\alpha = c(\{t_1, t_2\})$, the functions $f_{t_1} \restriction X_\alpha$ and $f_{t_2} \restriction X_\alpha$ are distinct elements of F_α. Now, let μ satisfy the partition relation

$$\mu \to (\kappa)^2_{\mathrm{cf}(\lambda)}.$$

By the Erdös-Rado theorem we have $(2^\gamma)^+ \to (\gamma^+)^2_\gamma$ for all γ, so we may assume

$$\mu \leq \nu = \max\{(2^\kappa)^+, (2^{\mathrm{cf}(\lambda)})^+\}.$$

Of course $\nu < 2^\lambda$, so $\mathcal{F}$ does indeed have size ν subsets.

We will now show that size μ (and therefore size ν) subsets of $\mathcal{F} = \{f_t : t \in T\}$ are unbounded in $\langle {}^\lambda\kappa, \leq\rangle$. Let T' be an arbitrary size μ subset of T. Since

$$c \restriction [T']^2 : [T']^2 \to \mathrm{cf}(\lambda)$$

and $|T'| = \mu$, we may fix a size κ subset T'' of T' that is monochromatic with respect to $c \restriction [T']^2$. Let $\alpha < \mathrm{cf}(\lambda)$ be the unique color assigned to all pairs from T''. The functions $f_t \restriction X_\alpha$ for $t \in T''$ are distinct elements of F_α. Hence, $\{f_t \restriction X_\alpha : t \in T''\}$ is an unbounded family of functions from X_α to κ. Thus, $\{f_t : t \in T''\}$ is unbounded in $\langle {}^\lambda\kappa, \leq \rangle$. □

We will now present a different way to understand functions from λ to $\kappa < \lambda$ where λ is singular. The following is a souped-up version of part 1) of Proposition III.9, and it is the natural way to show $\mathrm{cf}\,\langle {}^\lambda\kappa, \leq \rangle$ is large using PCF theory:

Lemma III.12. *Let $\lambda' \leq \lambda$ be infinite cardinals. Let $\kappa < \lambda$ be regular. Let f be a function that maps elements of λ' to regular cardinals in the interval $[\kappa, \lambda]$. Also assume*

$$(\forall \alpha < \lambda')\, f(\alpha)^\kappa \leq \lambda.$$

Then there exists a morphism from $\langle {}^\lambda\kappa, \leq \rangle$ to $\langle \prod_{\alpha < \lambda'} f(\alpha), \leq \rangle$.

Proof. This is very similar to case 1) of Proposition III.9. The point is that for each $\alpha < \lambda'$, there is a morphism from $\langle {}^\lambda\kappa, \leq \rangle$ to $\langle f(\alpha), \leq \rangle$, and these can all be combined together. □

Recall from PCF theory that given $\lambda' \in [\mathrm{cf}(\lambda), \lambda)$ with λ singular, $\mathrm{pp}_{\lambda'}(\lambda)$ is the supremum of all cofinalities of ultraproducts of sets of regular cardinals $A \subseteq \lambda$ satisfying $|A| \leq \lambda'$. The fact that this definition involves domination mod ultrafilters rather than everywhere domination is irrelevant because of the following:

1) the sets A in the definition can be assumed to be *progressive* ($|A| < \min(A)$);

2) if A is any progressive set of regular cardinals, then

$$\max \mathrm{pcf}(A) = \mathrm{cf}\,\langle \prod A, \leq \rangle.$$

For a proof of 2), see Theorem 3.4.21 in [23]. On the other hand, the restriction $\lambda' < \lambda$ instead of merely $\lambda' \leq \lambda$ is unfortunate for our situation. The following shows the importance of the $\mathrm{pp}_{\lambda'}(\lambda)$ function:

Fact III.13. *Let λ be a singular cardinal. Let $\lambda' \in [\mathrm{cf}(\lambda), \lambda)$ be a cardinal and assume $(\forall \sigma < \lambda)\, \sigma^{\lambda'} < \lambda$. Also assume one of the following:*

1) λ is not a fixed point of the aleph function;

2) $\mathrm{cf}(\lambda) > \omega$.

Then $\mathrm{pp}_{\lambda'}(\lambda) = \lambda^{\lambda'}$.

Proof. See Theorems 9.1.1 and 9.1.3 of [23]. □

With this fact, we get another proof that $\mathrm{cf}\,\langle {}^{\lambda}\kappa, \leq\rangle = 2^{\lambda}$ in almost all instances in which λ is a singular strong limit cardinal:

Proposition III.14. *Let λ be a singular cardinal, $\lambda' \in [\mathrm{cf}(\lambda), \lambda)$ be a cardinal, and $\kappa < \lambda$ be regular. Let $\nu = \max\{\lambda', \kappa\}$ and assume $(\forall \sigma < \lambda)\, \sigma^{\nu} < \lambda$. Also assume one of the following:*

1) λ is not a fixed point of the aleph function;

2) $\mathrm{cf}(\lambda) > \omega$.

Then $\mathrm{cf}\,\langle {}^{\lambda}\kappa, \leq\rangle \geq \lambda^{\lambda'}$. In particular, if λ is also a strong limit cardinal, and $\lambda' = \mathrm{cf}(\lambda)$ then

$$\mathrm{cf}\,\langle {}^{\lambda}\kappa, \leq\rangle = 2^{\lambda}.$$

Proof. It suffices to prove only the claim $\mathrm{cf}\,\langle {}^{\lambda}\kappa, \leq\rangle \geq \lambda^{\lambda'}$, because if λ is a strong limit cardinal then $\lambda^{\mathrm{cf}(\lambda)} = 2^{\lambda}$. Since $(\forall \sigma < \lambda)\, \sigma^{\lambda'} < \lambda$ and we are assuming either 1) or 2), by Fact III.13 we have

$$\mathrm{pp}_{\lambda'}(\lambda) = \lambda^{\lambda'}.$$

We now must show

$$\mathrm{cf}\,\langle {}^{\lambda}\kappa, \leq\rangle \geq \mathrm{pp}_{\lambda'}(\lambda).$$

It suffices to show that for an arbitrary set $A \subseteq \lambda$ of regular cardinals satisfying $|A| \leq \lambda'$ that

$$\mathrm{cf}\,\langle {}^{\lambda}\kappa, \leq\rangle \geq \max \mathrm{pcf}(A).$$

Fix such an A. Without loss of generality, by deleting an initial segment of A we may assume $A \subseteq [\kappa, \lambda)$. Of course,

$$\mathrm{cf}\,\langle \prod A, \leq\rangle \geq \max \mathrm{pcf}(A).$$

In fact, this is an equality when we assume A is progressive, but never mind this. Since $A \subseteq [\kappa, \lambda)$ and $(\forall \sigma < \lambda)\, \sigma^{\kappa} < \lambda$, applying Lemma III.12 we get that there is a morphism from $\langle {}^{\lambda}\kappa, \leq\rangle$ to $\langle \prod A, \leq\rangle$. Hence,

$$\mathrm{cf}\,\langle {}^{\lambda}\kappa, \leq\rangle \geq \mathrm{cf}\,\langle \prod A, \leq\rangle,$$

and we are done. □

3.5 An Independent Family of Borel Functions

We will now give a proof that $\mathrm{cf}\,\mathcal{B}_{\omega_1}(\omega, \leq) = 2^{\omega}$. We do this by constructing a size 2^{ω} family of Borel functions from ${}^{\omega}\omega$ to ω that is ω^{+}-independent, which is certainly sufficient. Indeed, we may easily convert the functions produced by the appropriate instance of Theorem II.26 into Borel functions from ${}^{\omega}\omega$ to ω.

To see this, let

$$\Lambda := \{(S, g) : S \in [\mathcal{P}(\omega)]^{\omega}, g : S \to \omega\}.$$

For each $A \subseteq \omega$, define $f_A : \Lambda \to \omega$ by

$$f_A(S, g) := \begin{cases} g(A) & \text{if } A \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\mathcal{F} := \{f_A : A \subseteq \omega\}.$$

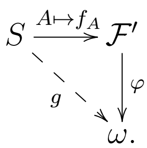
We will first show that $\mathcal{F}$ is ω^+-independent. That is,

$$(\forall \mathcal{F}' \in [\mathcal{F}]^\omega)(\forall \varphi : \mathcal{F}' \to \omega)(\exists x \in \Lambda)(\forall f \in \mathcal{F}')\, f(x) = \varphi(f).$$

Pick any $\mathcal{F}' \in [\mathcal{F}]^\omega$ and $\varphi : \mathcal{F}' \to \omega$. Let $S \in [\mathcal{P}(\omega)]^\omega$ be the set

$$S := \{A \subseteq \omega : f_A \in \mathcal{F}'\}.$$

Let $g : S \to \omega$ cause the following diagram to commute:



Let $x = (S, g)$. Then certainly

$$(\forall f_A \in \mathcal{F}')\, f_A(x) = f_A(S, g) = g(A) = \varphi(f_A).$$

Hence, $\mathcal{F}$ is ω^+-independent.

Now, by the definition of the functions f_A, we see that there is a nicely definable bijection $\eta : {}^\omega\omega \to \Lambda$ such that each function $\tilde{f}_A : {}^\omega\omega \to \omega$ defined by

$$\tilde{f}_A(x) := f_A(\eta(x))$$

is Borel. Hence,

$$\tilde{\mathcal{F}} := \{\tilde{f}_A : A \subseteq \omega\}$$

is an ω^+-independent size 2^ω family of Borel functions from ${}^\omega\omega$ to ω. Applying Proposition II.3 to the family $\tilde{\mathcal{F}}$ with $\mu = \omega$, we see that $\operatorname{cf} \mathcal{B}_{\omega_1}(\omega, \leq) = 2^\omega$.

Indeed, the functions $\tilde{f}_A$ are low down in the Baire hierarchy, so we have $\operatorname{cf} \mathcal{B}_\alpha(\omega, \leq) = 2^\omega$ for all but very small $\alpha \leq \omega_1$. We will not fret now about at which α this happens, because in Chapter IV we will prove that $\mathcal{B}_0(\omega, \leq) = \mathfrak{d}$, and at the beginning of Chapter V we will prove that $\mathcal{B}_\alpha(\omega, \leq) = 2^\omega$ for all $\alpha \geq 1$ in a way that provides much more information.

Now, if $2^\mathfrak{b} = 2^\omega$, by applying Theorem II.26 we get a $\mathfrak{b}^+$-independent size 2^ω family of functions from ${}^\omega\omega$ to ${}^\omega\omega$, and therefore each size $\mathfrak{b}$ subset is unbounded with respect to $\leq^*$. However, it is not clear how to convert that into a family of Borel functions. The problem is the corresponding definition of Λ would involve $[\mathcal{P}(\omega)]^\mathfrak{b}$, and so there should be no "nice" way to biject ${}^\omega\omega$ with Λ. Indeed, we see no easy way to prove that $\mathcal{B}_{\omega_1}({}^\omega\omega, \leq^*) = 2^\omega$.

CHAPTER IV

Impossibility of Coding by Continuous Functions

Consider the poset $\mathcal{B}_0(\omega, \leq)$ of continuous functions from ${}^{\omega}\omega$ to ω ordered by everywhere domination. The purpose of this chapter is to prove that $\operatorname{cf} \mathcal{B}_0(\omega, \leq) = \mathfrak{d}$ and discuss related problems. Combining this with the fact that $\mathfrak{d} < 2^{\omega}$ is consistent with ZFC, we conclude that ZFC *cannot* prove the following: for each $A \subseteq \omega$, Alice can construct a continuous function $f : {}^{\omega}\omega \to \omega$ such that if $g : {}^{\omega}\omega \to \omega$ is a continuous function which everywhere dominates f, then Bob can guess A from g using only countably many guesses. This is an *impossibility of coding* result. The combinatorial core of this chapter is that if we let $\mathcal{W}$ denote the set of well-founded subtrees of ${}^{<\omega}\omega$, then $\operatorname{cf} \langle \mathcal{W}, \subseteq \rangle = \mathfrak{d}$. This in turn follows from there existing a morphism from $\langle B_{-}^{\omega_1}, B_{+}^{\omega_1}, B^{\omega_1} \rangle$ (which we will define soon) to $\langle \mathcal{W}, \subseteq \rangle$. This chapter is not needed to understand the chapters which follow.

4.1 Well-founded Trees and Continuous Functions

Recall from Observation I.19 that

$$\mathfrak{d} \leq \operatorname{cf} \mathcal{B}_0({}^{\omega}\omega, \leq^*) \leq \operatorname{cf} \mathcal{B}_0(\omega, \leq) \leq 2^{\omega}.$$

Within this chapter, we will show that

$$\operatorname{cf}\mathcal{B}_0(\omega,\leq)\leq\mathfrak{d},$$

which will imply

$$\mathfrak{d}=\operatorname{cf}\mathcal{B}_0({}^{\omega}\omega,\leq^*)=\operatorname{cf}\mathcal{B}_0(\omega,\leq).$$

To be more concise in this chapter, we make the following two definitions:

Definition IV.1. Given $\alpha\leq\omega_1$, let $\mathcal{W}_\alpha$ be the set of well-founded subtrees of ${}^{<\omega}\omega$ of rank $<\alpha$. Let $\mathcal{W}:=\mathcal{W}_{\omega_1}$ be the set of all well-founded subtrees of ${}^{<\omega}\omega$.

Definition IV.2. Given $\alpha\leq\omega_1$, let B_-^α be the set of all functions from ${}^{<\omega}\omega$ to α, let B_+^α be the set of all functions from ${}^{<\omega}\omega$ to $[\alpha]^{<\omega}$, and let $B^\alpha\subseteq B_-^\alpha\times B_+^\alpha$ be defined by

$$fB^\alpha g\text{ iff }(\forall t\in{}^{<\omega}\omega)\,f(t)\in g(t).$$

In the definition above, we chose to use ${}^{<\omega}\omega$ as the domain of the functions instead of ω so that later we will not fuss with bijections between ${}^{<\omega}\omega$ and ω.

Temporarily fixing α that satisfies $\omega<\alpha<\omega_1$, we summarize in the following diagram the morphisms whose existence is either self-evident or we will prove in this chapter. A one-sided arrow represents the existence of a morphism, and a two-sided arrow represents the existence of a morphism in each direction.

$$\begin{array}{ccccccc}
\mathcal{B}_0(\omega,\leq) & \longleftrightarrow & \langle\mathcal{W},\subseteq\rangle & \longleftarrow & \langle B_-^{\omega_1},B_+^{\omega_1},B^{\omega_1}\rangle & & \\
 & & & & \downarrow & & \\
 & & \langle\mathcal{W}_\alpha,\subseteq\rangle & \longleftrightarrow & \langle B_-^{\alpha},B_+^{\alpha},B^{\alpha}\rangle & \longleftrightarrow & \langle{}^{\omega}\omega,\leq\rangle
\end{array}$$

The key result is that there exists a morphism from $\langle B_-^{\omega_1},B_+^{\omega_1},B^{\omega_1}\rangle$ to $\langle\mathcal{W},\subseteq\rangle$. This, combined with the fact that $||\langle B_-^{\omega_1},B_+^{\omega_1},B^{\omega_1}\rangle||=\mathfrak{d}$, implies that $\operatorname{cf}\mathcal{B}_0(\omega,\leq)\leq\mathfrak{d}$. We see no immediate reason for there to be a morphism from $\langle\mathcal{W},\subseteq\rangle$ to $\langle B_-^{\omega_1},B_+^{\omega_1},B^{\omega_1}\rangle$, but we have not explicitly ruled out the possibility.

We will now look closely at how continuous functions from ${}^{\omega}\omega$ to ω are specified.

Definition IV.3. A *barrier* is a set $S \subseteq {}^{<\omega}\omega$ satisfying

$$(\forall x \in {}^{\omega}\omega)(\exists! l \in \omega)\, x \restriction l \in S.$$

Proposition IV.4. *A function $f : {}^{\omega}\omega \to \omega$ is continuous iff there exist a barrier $S \subseteq {}^{<\omega}\omega$ and a function $\tilde{f} : S \to \omega$ satisfying*

$$(\forall x \in {}^{\omega}\omega)(\forall l \in \omega)[x \restriction l \in S \Rightarrow f(x) = \tilde{f}(x \restriction l)].$$

Proof. The $(\Leftarrow)$ direction is clear. For the $(\Rightarrow)$ direction, suppose that $f : {}^{\omega}\omega \to \omega$ is continuous. This implies that for each $x \in {}^{\omega}\omega$, there is some shortest finite initial segment s_x of x such that for all $y \in {}^{\omega}\omega$ extending s_x, $f(x) = f(y)$. Let

$$S := \{s_x : x \in {}^{\omega}\omega\}$$

and $\tilde{f} : S \to \omega$ be defined by

$$\tilde{f}(s) := f(x) \text{ where } x \text{ satisfies } s = s_x.$$

The function $\tilde{f}$ is well-defined and the condition is satisfied. □

If $S \subseteq {}^{<\omega}\omega$ is a barrier, then the set of all initial segments of elements of S is a well-founded tree. Because of this, one might expect that $\mathcal{B}_0(\omega, \leq)$ is related to $\langle \mathcal{W}, \subseteq \rangle$. This is indeed the case: as stated earlier, we will show that there are morphisms in both directions between $\mathcal{B}_0(\omega, \leq)$ and $\langle \mathcal{W}, \subseteq \rangle$.

Note that by associating well-founded trees to continuous functions from ${}^{\omega}\omega$ to ω, we may put these functions into a length ω_1 hierarchy based on the ranks of these trees.

4.2 The Morphisms

To begin, recall some basic definitions for maps from one poset to another:

Definition IV.5. Let $\langle P, \leq_P\rangle$ and $\langle Q, \leq_Q\rangle$ be two posets and let $i : P \to Q$ be a function. We say that i is *monotone* if

$$(\forall p_1, p_2 \in P)\, p_1 \leq_P p_2 \Rightarrow i(p_1) \leq_Q i(p_2),$$

i is *cofinal* if

$$(\forall q \in Q)(\exists p \in P)\, q \leq_Q i(p),$$

and i is *convergent* if it sends cofinal subsets of $\langle P, \leq_P\rangle$ to cofinal subsets of $\langle Q, \leq_Q\rangle$.

In the literature, what we call a convergent map is sometimes called a cofinal map (which is confusing). It is not hard to see that a map $i : P \to Q$ that is both monotone and cofinal is also convergent. When we introduced the concept of a morphism between posets in Section 1.2, we remarked that the existence of a convergent map is equivalent to the existence of a morphism (in the same direction).

We will now connect $\mathcal{B}_0(\omega, \leq)$ with $\langle \mathcal{W}, \subseteq\rangle$.

Proposition IV.6. *The map* Exit $: \mathcal{W} \to \mathcal{B}_0(\omega)$ *from* $\langle \mathcal{W}, \subseteq\rangle$ *to* $\mathcal{B}_0(\omega, \leq)$ *is both monotone and cofinal. Hence, there is a morphism from* $\langle \mathcal{W}, \subseteq\rangle$ *to* $\mathcal{B}_0(\omega, \leq)$*:*

$$\begin{array}{ccc} \mathcal{W} & \subseteq & \mathcal{W} \\ {\scriptstyle j}\uparrow & \Downarrow & \downarrow{\scriptstyle \mathrm{Exit}} \\ \mathcal{B}_0(\omega) & \leq & \mathcal{B}_0(\omega) \end{array}$$

Proof. Recall the definition of $\mathrm{Exit}(T)$ from Section 1.6. Certainly if $T_1 \subseteq T_2$, then $\mathrm{Exit}(T_1) \leq \mathrm{Exit}(T_2)$, which shows that Exit is monotone. To see that Exit is cofinal, fix a continuous $f \in \mathcal{B}_0(\omega)$. Let $S \subseteq {}^{<\omega}\omega$ be a barrier and $\tilde{f} : S \to \omega$ be a function specifying f as in Proposition IV.4. Let $S' \subseteq {}^{<\omega}\omega$ be a barrier such that each $s' \in S'$ extends some $s \in S$ and $|s'| \geq \tilde{f}(s)$. Let $j(f)$ be the set of all initial segments of elements of S'. It is not hard to see that $j(f)$ is a well-founded tree and $f \leq \mathrm{Exit}(j(f))$. Thus, Exit is cofinal. The pair $\langle j, \mathrm{Exit}\rangle$ is a morphism from $\langle \mathcal{W}, \subseteq\rangle$ to $\mathcal{B}_0(\omega, \leq)$. □

For completeness, let us state the complementary result:

Proposition IV.7. *There is a morphism from $\mathcal{B}_0(\omega, \leq)$ to $\langle \mathcal{W}, \subseteq\rangle$. Hence, there are morphisms in both directions between these relations.*

$$\begin{array}{ccccc} \mathcal{B}_0(\omega) & & \leq & & \mathcal{B}_0(\omega) \\ \scriptstyle\mathrm{Exit}\uparrow & & \Downarrow & & \downarrow\scriptstyle j \\ \mathcal{W} & & \subseteq & & \mathcal{W} \end{array}$$

Proof. Let $j : \mathcal{B}_0(\omega) \to \mathcal{W}$ be defined as in the proof of the proposition above. It is routine to verify that indeed $\langle \mathrm{Exit}, j\rangle$ is a morphism. □

The following characterization of the (ordinary) dominating number is more suitable for handling well-founded trees:

Proposition IV.8. *Let X and Y be any two countably infinite sets. Then $\mathfrak{d}$ is the smallest cardinality of a family $\mathcal{A}$ of functions from X to $[Y]^{<\omega}$ such that for each $f : X \to Y$, there is some $g \in \mathcal{A}$ satisfying $(\forall x \in X) f(x) \in g(x)$.*

Proof. Without loss of generality $X = Y = \omega$. Given a set $\mathcal{A}$ satisfying the property in the statement of this proposition,

$$\{n \mapsto \max f(n) : f \in \mathcal{A}\} \subseteq {}^{\omega}\omega$$

is cofinal in $\langle {}^{\omega}\omega, \leq\rangle$. Conversely, given a set $\mathcal{D}$ cofinal in $\langle {}^{\omega}\omega, \leq\rangle$,

$$\{n \mapsto \{m : m \leq f(n)\} : f \in \mathcal{D}\}$$

satisfies the property in the statement of this proposition. □

Here is the morphism version of the proposition above, using the particular sets X and Y that we will use for the main combinatorial result of this section:

Proposition IV.9. *Fix α satisfying $\omega \leq \alpha < \omega_1$. There are morphisms in both directions between $\langle {}^{\omega}\omega, \leq \rangle$ and $\langle B^{\alpha}_{-}, B^{\alpha}_{+}, B^{\alpha} \rangle$.*

Proof. This is routine using the ideas in the proof of the proposition above. □

The following is a curious result that builds upon the idea in Proposition IV.8:

Proposition IV.10. *For each $n \in \omega$, $\max\{\omega_n, \mathfrak{d}\}$ is the smallest cardinality of a family $\mathcal{A}$ of functions from ω to $[\omega_n]^{<\omega}$ such that for each $f : \omega \to \omega_n$, there is some $g \in \mathcal{A}$ satisfying $(\forall n \in \omega) f(n) \in g(n)$.*

Proof. We will prove this by induction. The $n = 0$ case follows by Proposition IV.8. For the successor step, assume the proposition holds for some fixed $n \in \omega$. We will show that it holds for $n + 1$. Let λ be the smallest cardinality of a family $\mathcal{B}$ of functions from ω to $[\omega_{n+1}]^{<\omega}$ such that for each $f : \omega \to \omega_{n+1}$, there is some $g \in \mathcal{B}$ satisfying $(\forall n \in \omega) f(n) \in g(n)$. By considering the constant functions from ω to ω_{n+1}, we see that $\lambda \geq \omega_{n+1}$. By considering the functions from ω to $[\omega]^{<\omega} \subseteq [\omega_{n+1}]^{<\omega}$, we see that $\lambda \geq \mathfrak{d}$. Thus, we have $\lambda \geq \max\{\omega_{n+1}, \mathfrak{d}\}$.

For the other direction, we will use the induction hypothesis. That is, for each $\alpha < \omega_{n+1}$, there is some family $\mathcal{A}_\alpha$ of cardinality $\max\{\omega_n, \mathfrak{d}\}$ of functions from ω to $[\alpha]^{<\omega}$ such that for each $f : \omega \to \alpha$, there is some $g \in \mathcal{A}_\alpha$ satisfying $(\forall n \in \omega) f(n) \in g(n)$. Let $\mathcal{A} := \bigcup_{\alpha < \omega_{n+1}} \mathcal{A}_\alpha$. Given an $f : \omega \to \omega_{n+1}$, there is some $\alpha < \omega_{n+1}$ such that $\mathrm{Im}(f) \subseteq \alpha$, so there is some $g \in \mathcal{A}_\alpha \subseteq \mathcal{A}$ satisfying $(\forall n \in \omega) f(n) \in g(n)$. Doing an easy calculation, we see that

$$|\mathcal{A}| = \sum_{\alpha < \omega_{n+1}} \max\{\omega_n, \mathfrak{d}\} = \max\{\omega_{n+1}, \max\{\omega_n, \mathfrak{d}\}\} = \max\{\omega_{n+1}, \mathfrak{d}\}.$$

Hence, $\lambda \leq \max\{\omega_{n+1}, \mathfrak{d}\}$. □

We will now prove the main combinatorial result of this section:

Proposition IV.11. *Fix α satisfying $\omega < \alpha \leq \omega_1$. There is a morphism $\langle \phi_-, \phi_+ \rangle$ from $\langle B^\alpha_-, B^\alpha_+, B^\alpha \rangle$ to $\langle \mathcal{W}_\alpha, \subseteq \rangle$:*

$$\begin{array}{ccc} B^\alpha_- & B^\alpha & B^\alpha_+ \\ \phi_- \big\uparrow & \big\Downarrow & \big\downarrow \phi_+ \\ \mathcal{W}_\alpha & \subseteq & \mathcal{W}_\alpha. \end{array}$$

Proof. Given a well-founded tree $T \subseteq {}^{<\omega}\omega$, each element of T has a rank. Let us use the convention for this proof that leaf nodes have rank 1. This allows us to say that elements of ${}^{<\omega}\omega - T$ have rank 0 (which we will do). Given a well-founded tree $T \subseteq {}^{<\omega}\omega$, let $\phi_-(T) : {}^{<\omega}\omega \to \alpha$ be the function that assigns each element of ${}^{<\omega}\omega$ its rank.

Fix a function $g : {}^{<\omega}\omega \to [\alpha]^{<\omega}$. We will soon define the well-founded tree $T = \phi^+(g) \subseteq {}^{<\omega}\omega$. First, we will define a function $h : {}^{<\omega}\omega \to \alpha$ such that for all $t_1, t_2 \in {}^{<\omega}\omega$ satisfying $t_1 \sqsubseteq t_2$ and $t_1 \neq t_2$, either $h(t_1) = h(t_2) = 0$ or $h(t_1) > h(t_2)$. Given such an h, it follows that $\{t \in {}^{<\omega}\omega : h(t) > 0\}$ is a well-founded tree, and this will be our T. Let $h(t)$ be defined by recursion on the length of t as follows:

1) $h(\emptyset) := \max g(\emptyset)$;

2) $h(t^\frown n) := \begin{cases} 0 & \text{if } h(t) = 0, \\ \max\{\beta \in g(t^\frown n) : \beta < h(t)\} & \text{otherwise.} \end{cases}$

The function h is well-defined (we use the convention that $\max \emptyset = 0$). It is also easy to see that h satisfies the desired condition, so T is indeed well-founded.

We have now defined ϕ_- and ϕ_+. All that remains is to verify that indeed

$$(\forall T_1 \in \mathcal{W}_\alpha)(\forall g \in B^\alpha_+)\, \phi_-(T_1) B^\alpha g \Rightarrow T_1 \subseteq \phi_+(g).$$

Fix any well-founded tree $T_1 \subseteq {}^{<\omega}\omega$. Let $f = \phi_-(T_1)$. That is, f is the rank function of T_1. Fix any function $g : {}^{<\omega}\omega \to [\alpha]^{<\omega}$ satisfying $(\forall t \in {}^{<\omega}\omega) f(t) \in g(t)$. Let $T_2 = \phi_+(g)$. We will show that $T_1 \subseteq T_2$, and then the proof will be complete.

Let $h : {}^{<\omega}\omega \to \alpha$ be the function defined from g as above. If we show

$$(\forall t \in {}^{<\omega}\omega) f(t) \leq h(t),$$

then we will have $T_1 \subseteq T_2$, because $T_1 = \{t \in {}^{<\omega}\omega : f(t) > 0\}$ and $T_2 = \{t \in {}^{<\omega}\omega : h(t) > 0\}$. We will show this by induction on the length of t. The base case is simple: $f(\emptyset) \leq \max g(\emptyset) =: h(\emptyset)$, because $f(\emptyset) \in g(\emptyset)$. For the successor step, assume $f(t) \leq h(t)$. Fix $n \in \omega$. We will show $f(t^\frown n) \leq h(t^\frown n)$. There are two cases. The first case is that $f(t) = 0$, which implies $f(t^\frown n) = 0$, so certainly $f(t^\frown n) \leq h(t^\frown n)$. The other case is that $f(t) > 0$. When this happens, $f(t^\frown n) < f(t)$. Combining this with the induction hypothesis that $f(t) \leq h(t)$ gives us that $f(t^\frown n) < h(t)$. Since also $f(t^\frown n) \in g(t^\frown n)$, we have

$$f(t^\frown n) \leq \max\{\beta \in g(t^\frown n) : \beta < h(t)\} = h(t^\frown n).$$

The proof is now complete. $\square$

For completeness, we prove a partially complementary result:

Proposition IV.12. *Fix α satisfying $\omega < \alpha < \omega_1$. There is a morphism from $\langle \mathcal{W}_\alpha, \subseteq \rangle$ to $\langle B^\alpha_-, B^\alpha_+, B^\alpha \rangle$.*

Proof. We showed in Proposition IV.9 that there is a morphism from $\langle {}^\omega\omega, \leq \rangle$ to $\langle B^\alpha_-, B^\alpha_+, B^\alpha \rangle$. Hence, since morphisms can be composed together, it suffices to show that there is one from $\langle \mathcal{W}_\alpha, \subseteq \rangle$ to $\langle {}^\omega\omega, \leq \rangle$:

$$\begin{array}{ccc} \mathcal{W}_\alpha & \subseteq & \mathcal{W}_\alpha \\ {\scriptstyle i}\uparrow & \Downarrow & \downarrow{\scriptstyle j} \\ {}^{\omega}\omega & \leq & {}^{\omega}\omega \end{array}$$

Given $f \in {}^\omega\omega$, let $i(f) \in \mathcal{W}_{\omega+1} \subseteq \mathcal{W}_\alpha$ be a tree which contains all sequences of the form

$$\langle n \rangle^\frown \overbrace{\langle 0, ..., 0 \rangle}^{f(n)}$$

for $n \in \omega$. Given $T \in \mathcal{W}_\alpha$, let $j(T) \in {}^\omega\omega$ be the function such that for each $n \in \omega$, $j(T)(n)$ is the largest k satisfying

$$\langle n \rangle^\frown \overbrace{\langle 0, ..., 0 \rangle}^{k} \in T.$$

The pair $\langle i, j \rangle$ is the desired morphism. □

Incidentally, Proposition IV.11 was discovered by first looking at whether each well-founded tree $T_1 \subseteq {}^{<\omega}\omega$ in the Sacks forcing extension is a subset of one such tree the ground model (with the hope of showing $\mathrm{cf}\,\mathcal{B}_0(\omega, \leq) < 2^\omega$ in the *Sacks model*). This was shown to be the case by using the *Sacks property*. That is, the ground model can guess the rank of each node of T_1. The Sacks property was then replaced with the weaker property of being ${}^\omega\omega$-bounding (which we will define in the next section). At this point, no other facts about the forcing were used. Then, the combinatorics of what was "really going on" was extracted. This is an example of forcing being used to discover a ZFC theorem.

4.3 Applications

Proposition IV.11 immediately allows us to prove some interesting results.

Theorem IV.13. *For each α satisfying $\omega < \alpha \leq \omega_1$, $\mathrm{cf}\,\langle \mathcal{W}_\alpha, \subseteq \rangle = \mathfrak{d}$.*

Proof. Let $\mathcal{D}$ be a size $\mathfrak{d}$ family of functions from ${}^{<\omega}\omega$ to $[\alpha]^{<\omega}$ such that for each $f : {}^{<\omega}\omega \to \alpha$, there is some $g \in \mathcal{D}$ satisfying $(\forall t \in {}^{<\omega}\omega) f(t) \in g(t)$. Let ϕ_+ be the function given by Proposition IV.11. Then

$$\mathcal{A} := \{\phi_+(g) : g \in \mathcal{D}\}$$

is cofinal in $\langle \mathcal{W}_\alpha, \subseteq\rangle$ of size at most $\mathfrak{d}$. On the other hand, since $\omega < \alpha$, it is clear that $\mathrm{cf}\,\langle \mathcal{W}_\alpha, \subseteq\rangle \geq \mathfrak{d}$. For a formal explanation of this, we showed in Proposition IV.12 that there is a morphism from $\langle \mathcal{W}_\alpha, \subseteq\rangle$ to $\langle {}^\omega\omega, \leq\rangle$. □

We can now compute $\mathrm{cf}\,\mathcal{B}_0(\omega, \leq)$ and $\mathrm{cf}\,\mathcal{B}_0({}^\omega\omega, \leq^*)$ as promised.

Corollary IV.14. $\mathrm{cf}\,\mathcal{B}_0(\omega, \leq) = \mathrm{cf}\,\mathcal{B}_0({}^\omega\omega, \leq^*) = \mathfrak{d}$.

For the skeptic who questions the need for the generality given by all these morphisms, we state some practical results which make use of them. Indeed, it is good practice to state results in terms of morphisms whenever possible, because this generality is required for certain proofs.

Recall the following:

Definition IV.15. Let M and N be transitive models of ZF with $M \subseteq N$. We say that N is *${}^\omega\omega$-bounding* over M if $({}^\omega\omega)^M$ is cofinal in $\langle ({}^\omega\omega)^N, \leq\rangle$.

The morphisms we constructed provide useful information when V is ${}^\omega\omega$-bounding over M and $\omega_1^M = \omega_1$:

Theorem IV.16. *Let M be a transitive model of ZF such that V is ${}^\omega\omega$-bounding over M. Assume also that $\omega_1^M = \omega_1$. Given any well-founded tree $T_1 \subseteq {}^{<\omega}\omega$, there is some well-founded tree $T_2 \subseteq {}^{<\omega}\omega$ in M satisfying $T_1 \subseteq T_2$.*

Proof. Let $T_1 \subseteq {}^{<\omega}\omega$ be an arbitrary well-founded tree. Fix its rank $\alpha < \omega_1$. By combining Proposition IV.11 and Proposition IV.9, we get a morphism $\langle i, j\rangle$ from

$\langle {}^\omega\omega, \leq \rangle$ to $\langle \mathcal{W}_{\alpha+1}, \subseteq \rangle$:

$$\begin{array}{ccc} {}^\omega\omega & \leq & {}^\omega\omega \\ {\scriptstyle i}\uparrow & \Downarrow & \downarrow{\scriptstyle j} \\ \mathcal{W}_{\alpha+1} & \subseteq & \mathcal{W}_{\alpha+1}. \end{array}$$

Since V is ${}^\omega\omega$-bounding over M, there is some $g \in ({}^\omega\omega)^M$ satisfying $i(T_1) \leq g$. Let $T_2 := j(g)$. Since $\langle i, j \rangle$ is a morphism, $T_1 \subseteq T_2$. Once we show $T_2 \in M$, we will be done.

Being a model of ZF, M has its own version

$$j^M : ({}^\omega\omega)^M \to (\mathcal{W}_{\alpha+1})^M$$

of the function j. The function j is certainly Borel, which gives us enough absoluteness to conclude that $j \restriction M = j^M$. Hence,

$$j(g) = (j \restriction M)(g) = j^M(g),$$

so $T_2 \in M$. □

The $\omega_1^M = \omega_1$ hypothesis in the theorem above is certainly necessary, because ω_1^M is the supremum of the set of ranks of well-founded subtrees of ${}^{<\omega}\omega$ in M, and $T_1 \subseteq T_2$ implies $\text{rank}(T_1) \leq \text{rank}(T_2)$.

Corollary IV.17. *Let M be a transitive model of ZF such that V is ${}^\omega\omega$-bounding over M. Suppose also that $\omega_1^M = \omega_1$. Then for each Borel code c_1 for a continuous function from ${}^\omega\omega$ to ω, there is a Borel code c_2 in M for a continuous function such that the function coded by c_2 everywhere dominates the function coded by c_1.*

Proof. Let c_1 be a Borel code for a continuous function f_1 from ${}^\omega\omega$ to ω. By Proposition IV.6, there is a map $i : \mathcal{W} \to \mathcal{B}_0(\omega)$ from $\langle \mathcal{W}, \subseteq \rangle$ to $\mathcal{B}_0(\omega, \leq)$ that is monotone and cofinal. Since i is cofinal, fix a well-founded tree $T_1 \subseteq {}^{<\omega}\omega$ satisfying $f_1 \leq i(T_1)$.

By the previous theorem, fix a well-founded tree $T_2 \subseteq {}^{<\omega}\omega$ in M such that $T_1 \subseteq T_2$. Since i is monotonic, $i(T_1) \leq i(T_2)$.

Now, since M is a model of ZF, it has its own version of i, which we denote by i^M. Within M, there is a Borel code c_2 for $i^M(T_2)$. In V, c_2 codes $f_2 := i(T_2)$. We now have

$$f_1 \leq i(T_1) \leq i(T_2) = f_2,$$

and the proof is complete. □

The cost of not using morphisms is having multiple proofs with duplicated combinatorial content. That is, if we proved both of the above theorems directly, then the content of Proposition IV.11 would be written twice.

4.4 Nonexistence of Nicely Definable Morphisms

We close this chapter with a negative result: there cannot exist a "nicely" definable morphism $\langle \phi_-, \phi_+ \rangle$ from $\langle {}^\omega\omega, \leq \rangle$ to $\langle B_-^{\omega_1}, B_+^{\omega_1}, B^{\omega_1} \rangle$. For example, if we assume $L(\mathbb{R})$ satisfies AD, then since there cannot be an injection from ω_1 into ${}^\omega\omega$ in $L(\mathbb{R})$ and (ω_1 is regular)$^{L(\mathbb{R})}$, there cannot exist such a morphism $\langle \phi_-, \phi_+ \rangle$ where $\phi_-, \phi_+ \in L(\mathbb{R})$. In fact, an analysis of the proofs below show that there cannot exist a $\langle \phi_-, \phi_+ \rangle$ where $\phi_- \in L(\mathbb{R})$.

Proposition IV.18. *(ZF) Assume there is no injection from ω_1 into ${}^\omega\omega$ and ω_1 is regular. Then there is no morphism from $\langle {}^\omega\omega, \leq \rangle$ to $\langle \omega_1, \leq \rangle$.*

Proof. Assume ω_1 is regular. Let $\langle \phi_-, \phi_+ \rangle$ be a morphism from $\langle {}^\omega\omega, \leq \rangle$ to $\langle \omega_1, \leq \rangle$:

$$\begin{array}{ccc} {}^\omega\omega & \leq & {}^\omega\omega \\ {\scriptstyle \phi_-}\uparrow & \Downarrow & \downarrow{\scriptstyle \phi_+} \\ \omega_1 & \leq & \omega_1. \end{array}$$

We will construct (in ZF) an injection from ω_1 into ${}^\omega\omega$. It suffices to construct a size ω_1 set $A \subseteq \omega_1$ such that $\phi_- \restriction A$ is injective. First note that $|\mathrm{Im}(\phi_-)| = \omega_1$ because if not, then by the pigeon hole principle (since ω_1 is regular), there would be a single $g \in {}^\omega\omega$ such that $\phi_-(\alpha) = g$ for ω_1 many $\alpha < \omega_1$. Since $\langle \phi_-, \phi_+ \rangle$ is a morphism, this would imply that $\alpha \leq \phi_+(g)$ for ω_1 many $\alpha < \omega_1$, which is clearly impossible.

We may now inductively define $A := \{a_\alpha : \alpha < \omega_1\}$ as follows: let $a_0 := 0$. For each $\alpha > 0$, let $a_\alpha < \omega_1$ be the smallest ordinal such that $\phi_-(a_\alpha) \neq \phi_-(a_\beta)$ for all $\beta < \alpha$. We will never get stuck because $|\mathrm{Im}(\phi_-)| = \omega_1$. By construction, $\phi_- \restriction A$ is injective. $\square$

Proposition IV.19. *(ZF) Assume there is no injection from ω_1 into ${}^\omega\omega$ and ω_1 is regular. Then there is no morphism from $\langle {}^\omega\omega, \leq \rangle$ to $\langle B_-^{\omega_1}, B_+^{\omega_1}, B^{\omega_1} \rangle$.*

Proof. We will prove the contrapositive. Let $\langle \phi_-, \phi_+ \rangle$ be a morphism:

$$\begin{array}{ccc} {}^\omega\omega & \leq & {}^\omega\omega \\ \phi_- \uparrow & \Downarrow & \downarrow \phi_+ \\ B_-^{\omega_1} & B^{\omega_1} & B_+^{\omega_1}. \end{array}$$

There is also a morphism $\langle \psi_-, \psi_+ \rangle$ from $\langle B_-^{\omega_1}, B_+^{\omega_1}, B^{\omega_1} \rangle$ to $\langle \omega_1, \leq \rangle$ given by $\psi_-(\alpha) := (t \mapsto \alpha)$ and $\psi_+(g) := \sup \bigcup_{t \in {}^{<\omega}\omega} g(t)$:

$$\begin{array}{ccc} B_-^{\omega_1} & B^{\omega_1} & B_+^{\omega_1} \\ \psi_- \uparrow & \Downarrow & \downarrow \psi_+ \\ \omega_1 & \leq & \omega_1. \end{array}$$

By composing these morphisms together, we get one from $\langle {}^\omega\omega, \leq \rangle$ to $\langle \omega_1, \leq \rangle$. We now apply the proposition above to complete the proof. $\square$

CHAPTER V

Everywhere Domination Coding Theorems

In this chapter, we will see that $\mathcal{B}_\alpha(\omega,\le)$ for $\alpha \ge 1$ has a completely different nature than $\mathcal{B}_0(\omega,\le)$. First, we will show that while well-founded trees were the key to understanding $\mathcal{B}_0(\omega,\le)$, *clouds* are the key to understanding $\mathcal{B}_1(\omega,\le)$. Clouds allow us to convert the problem of computing $\operatorname{cf}\mathcal{B}_1(\omega,\le)$ into a problem that is more combinatorial. This quickly leads to the proof that $\operatorname{cf}\mathcal{B}_\alpha(\omega,\le) = 2^\omega$ for each $\alpha \ge 1$. The essential observation is that for each $a \in {}^\omega\omega$, if $g : {}^\omega\omega \to \omega$ is *any* function which satisfies $\operatorname{Exit}([[a]]) \le g$, then a is Δ^1_1 in a predicate for g. In particular, if g is Borel, then a is Δ^1_1 in any code for g. We may view this as an infinite coding result: Alice encodes her message $a \in {}^\omega\omega$ into the function $f = \operatorname{Exit}([[a]])$, and when an enemy steps in and produces a function g which satisfies $f \le g$, then Bob can guess a from g by making countably many guesses: guessing each real which is Δ^1_1 in a predicate for g.

The encoding $a \mapsto \operatorname{Exit}([[a]])$ we may call *vertical* coding. There is a different natural encoding scheme we may use: *horizontal* coding. With horizontal coding, we easily get a new proof that $\operatorname{cf}\operatorname{All}(\omega,\le) = 2^{2^\omega}$ by showing that for each $A \subseteq {}^\omega\omega$, there is a function $f : {}^\omega\omega \to \omega$ such that if $g : {}^\omega\omega \to \omega$ satisfies $f \le g$, then A is $\boldsymbol{\Delta}^1_1$ in a predicate for g. The two methods are incomparable in that they generalize in different but important ways (and so we must keep both methods). Unfortunately, there is not one single unifying coding theorem we may prove and then derive all related coding results from that. The contribution of this chapter is a general argument which can be enhanced in various ways, but all enhancements cannot be made simultaneously. We have taken the approach of presenting each argument in a self contained way at the expense of being slightly repetitive.

A desirable feature of our prototypical coding result is that it only requires

$$(\forall x \in ({}^{\omega}\omega)^{L[g]})\, f(x) \leq g(x)$$

instead of $f \leq g$. This generality is important because it gives rise to applications to weak distributivity laws for complete Boolean algebras. After we sufficiently understand $\mathcal{B}_\alpha(\omega, \leq)$ for $\alpha \geq 1$, we change gears to apply the arguments to combinatorial set theory. That is, we apply our coding arguments to functions from ${}^{\kappa}\lambda$ to κ for infinite cardinals κ and λ. In this context, we get the "main coding theorems" which quicky give us the implications for weak distributivity laws for complete Boolean algebras.

Specifically, if $\mathbb{B}$ is a complete Boolean algebra which is weakly (λ^ω, ω)-distributive for an infinite cardinal λ, then $\mathbb{B}$ is $(\lambda, 2)$-distributive. Next, if κ is a weakly compact cardinal, $\mathbb{B}$ is weakly $(2^\kappa, \kappa)$-distributive, and $\mathbb{B}$ is $(\alpha, 2)$ distributive for each $\alpha < \kappa$, then $\mathbb{B}$ is $(\kappa, 2)$-distributive. Finally, if $\mathbb{B}$ is weakly $(2^{\omega_1}, \omega_1)$-distributive, $\mathbb{B}$ is $(\omega, 2)$-distributive, and $1 \Vdash_{\mathbb{B}} (\omega_1 < \mathfrak{t})$, then $\mathbb{B}$ is $(\omega_1, 2)$-distributive.

5.1 Clouds and Baire Class One Functions

Recall that $\mathcal{B}_1(\omega, \leq)$ is the set $\mathcal{B}_1(\omega)$ of Baire class one functions from ${}^{\omega}\omega$ to ω ordered pointwise by $\leq$. That is, $\mathcal{B}_1(\omega)$ is the set of pointwise limits of continuous functions from ${}^{\omega}\omega$ to ω. There is an apparently easier to understand cofinal subset of $\mathcal{B}_1(\omega, \leq)$:

Definition V.1. $\mathcal{F}_1 \subseteq \mathcal{B}_1(\omega)$ is the set of all functions each of which is the pointwise maximum of an ω-sequence of elements of $\mathcal{B}_0(\omega)$.

This is indeed cofinal because if $g \in \mathcal{B}_1(\omega)$ is the pointwise limit of the sequence of continuous functions $\langle f_n : n \in \omega \rangle$, then

$$h(x) := \max\{f_n(x) : n \in \omega\}$$

is in $\mathcal{F}_1$ and $g \leq h$.

In fact, if we start with $\mathcal{B}_0(\omega)$ and alternate between taking pointwise maximums and pointwise minimums, then after ω_1 stages we will have precisely all Borel functions from ${}^{\omega}\omega$ to ω. This is because if $\langle f_n : n \in \omega \rangle$ is a sequence of functions from ${}^{\omega}\omega$ to ω and for each x the limit $\lim_{n\to\infty} f_n(x)$ exists, then for each x we have

$$\lim_{n\to\infty} f_n(x) = \max_n \min_{m\geq n} f_m(x) = \min_n \max_{m\geq n} f_m(x).$$

This shows that the hierarchy we get by alternating between taking maximums and minimums is closely related to the Baire hierarchy. For example, they are equal at limit stages.

We should point out that there is another way to construct the Baire hierarchy [8]. That is, first construct the smallest collection of filters on ω starting with the cofinite filter and closed under sums $\mathcal{V}$–$\sum_i \mathcal{U}_i$. Then the collection of Borel functions is the same as the collection of filter limits, using filters in this collection, of continuous functions.

The reason for introducing $\mathcal{F}_1$ is because it has a simple combinatorial characterization in terms of *clouds* which is useful for us. In the same way that well-founded trees were the right way to understand $\mathcal{B}_0(\omega, \leq)$ (Propositions IV.6 and IV.7), clouds are the right way to understand $\langle \mathcal{F}_1, \leq \rangle$ (and therefore $\operatorname{cf} \mathcal{B}_1(\omega, \leq)$). We use the convention that $\max \emptyset = 0$.

Definition V.2. A set $C \subseteq {}^{<\omega}\omega$ is called a *cloud* if for each $x \in {}^{\omega}\omega$,

$$\{l \in \omega : x \upharpoonright l \in C\}$$

is finite. The function $\operatorname{Rep}(C) : {}^{\omega}\omega \to \omega$ is defined by

$$\operatorname{Rep}(C)(x) := \max\{l : x \upharpoonright l \in C\}.$$

That is, a subset of ${}^{<\omega}\omega$ is a cloud if its intersection with each path through ${}^{<\omega}\omega$ is finite. The function $\operatorname{Rep}(C)$ ("Rep" for "Representation") outputs the greatest level at which x hits C. This can be generalized to handle functions from ${}^{\kappa}X$ to κ, where κ is a cardinal and X is a set (this is precisely Definition I.29 given in the introduction). Here is the promised characterization:

Proposition V.3. *A function $f : {}^{\omega}\omega \to \omega$ is in $\mathcal{F}_1$ iff there is a cloud $C \subseteq {}^{<\omega}\omega$ and a function $\tilde{f} : C \to \omega$ such that for all $x \in {}^{\omega}\omega$,*

$$f(x) = \max\{\tilde{f}(x \upharpoonright l) : x \upharpoonright l \in C\}.$$

Proof. First, if there is such a cloud C and a function $\tilde{f}$, then for each $c \in C$ define $f_c : {}^{\omega}\omega \to \omega$ to be the continuous function

$$f_c(x) := \begin{cases} \tilde{f}(c) & \text{if } x \sqsupseteq c, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that for each x, $f(x) = \max\{f_c(x) : c \in C\}$, and so $f \in \mathcal{F}_1$.

For the other direction, suppose $f \in \mathcal{F}_1$. Let $\langle f_n : n \in \omega \rangle$ be an ω-sequence of continuous functions such that for each x, $f(x) = \max\{f_n(x) : n \in \omega\}$. We may assume, without loss of generality, that for each x, $f_{n_1}(x) \leq f_{n_2}(x)$ whenever $n_1 \leq n_2$. For each $n \in \omega$, by Proposition IV.4 let $S_n \subseteq {}^{<\omega}\omega$ be a barrier and $\tilde{f}_n : S_n \to \omega$ be a function such that $f_n(x) = \tilde{f}_n(t)$ whenever x extends t and $t \in S_n$. We may also assume that S_{n_2} properly extends S_{n_1} whenever $n_1 < n_2$, by which we mean for all $x \in {}^{\omega}\omega$, the level where x hits S_{n_1} is strictly below the level where x hits S_{n_2}. Hence, the sets S_n are pairwise disjoint.

For each $n > 0$, define the set $S'_n \subseteq S_n$ as follows:

$$S'_n := \{c \in S_n : (\forall x \sqsupseteq c)\, f_n(x) > f_{n-1}(x)\}.$$

Note that also

$$S'_n = \{c \in S_n : (\exists x \sqsupseteq c)\, f_n(x) > f_{n-1}(x)\}.$$

Define $C := S_0 \cup \bigcup_{n>0} S'_n$. We claim that C is a cloud. Let $x \in {}^{\omega}\omega$ be arbitrary. We must show that x hits C at only finitely many places. If not, then by construction $\{f_n(x) : n \in \omega\}$ is unbounded, which contradicts the fact that f is well-defined at x. Hence, C is a cloud, and we may define $\tilde{f} : C \to \omega$ in the natural way: $\tilde{f}(c) := \tilde{f}_n(c)$ where n is the unique number satisfying $c \in S_n$. It is not difficult to check that $\tilde{f}$ is as desired. □

Notice in the construction above that $\tilde{f}(t_1) < \tilde{f}(t_2)$ for all $t_1, t_2 \in C$ with t_1 a proper initial segment of t_2. The collection of clouds itself has structure. There is a natural ω_1-length hierarchy into which all clouds may be placed.

Definition V.4. Given $\alpha < \omega_1$, a cloud C is an *α-cloud* if α is $\geq$ the rank of the well-founded tree that is the set $C \cup \{\emptyset\}$ ordered by end-extension.

Here we use the convention that leaf nodes have rank 0. Thus, if each $x \in {}^{\omega}\omega$ hits C at most 1 time, then C is a 1-cloud. The functions *represented* by clouds form a cofinal subset of $\mathcal{F}_1$ which is simpler to understand.

Proposition V.5. *For each $f \in \mathcal{F}_1$, there exists a cloud $C \subseteq {}^{<\omega}\omega$ satisfying $f \leq \mathrm{Rep}(C)$. Moreover, if f is specified by a cloud $C_f \subseteq {}^{<\omega}\omega$ and a function $\tilde{f} : C_f \to \omega$ as in the proposition above, then if C_f is an α-cloud, then C can be chosen to be an α-cloud.*

Proof. Let $C_f \subseteq {}^{<\omega}\omega$ and $\tilde{f} : C_f \to \omega$ specify f as in the proposition above. Assume C_f is an α-cloud. Without loss of generality, C_f is infinite. The idea of how to proceed is simple: we replace each node $c \in C_f$ with an appropriate set of nodes extending it. We must be careful to ensure the resulting cloud C is indeed an α-cloud.

First, let $e : \omega \to C_f$ be a bijection that respects the ordering on C_f by extension. That is, for all $n_1, n_2 \in \omega$, if $e(n_1) \sqsubseteq e(n_2)$, then $n_1 \leq n_2$. We may easily define a function $l : \omega \to \omega$ that is both strictly increasing and such that for all $n \in \omega$,

$$l(n) \geq \tilde{f}(e(n)).$$

Given such an l, define the function $S : \omega \to \mathcal{P}({}^{<\omega}\omega)$ as follows:

$$S(n) := \{c' \in {}^{l(n)}\omega : c' \sqsupseteq e(n)\}.$$

That is, $S(n)$ is the set of all extensions of $e(n)$ on level $l(n)$. We may now define C as follows:

$$C := \bigcup_{n \in \omega} S(n).$$

It is not difficult to see that C is a cloud. We have $f \leq \mathrm{Rep}(C)$ because $l(n) \geq \tilde{f}(e(n))$ for all $n \in \omega$. Moreover, since we were careful (by requiring e to be order respecting and l to be strictly increasing), the tree that is the set of elements of C ordered by end-extension has the same rank as the tree corresponding to C_f. Since C_f is an α-cloud, so is C. □

Each Baire class one function from ${}^{\omega}\omega$ to ω is $\leq$ one represented by an α-cloud for some $\alpha < \omega_1$. It can be shown that the hierarchy of functions represented by clouds does not collapse, in the sense that for each $\alpha < \omega_1$, there is some function represented by an α-cloud that is not $\leq$ any function represented by a β-cloud for $\beta < \alpha$. We will not dwell on this hierarchy, but instead focus on the very bottom level. The simplest (non-trivial) kind of cloud is a 1-cloud. We have an alternate characterization of functions represented by 1-clouds in terms of the Exit function of Definition I.28.

If $T \subseteq {}^{<\omega}\omega$ is well-founded, then $\mathrm{Exit}(T)$ is continuous. By Proposition IV.6, for each continuous $f : {}^{\omega}\omega \to \omega$ there is some well-founded $T \subseteq {}^{<\omega}\omega$ satisfying $f \leq \mathrm{Exit}(T)$. Dropping the requirement that T be well-founded we get precisely the functions represented by 1-clouds:

Proposition V.6. *Given a function $f : {}^{\omega}\omega \to \omega$, $f = \mathrm{Rep}(C)$ for some 1-cloud C iff $f = \mathrm{Exit}(T)$ for some tree $T \subseteq {}^{<\omega}\omega$.*

Proof. If f is represented by a 1-cloud C, then the set

$$T := \{t \in {}^{<\omega}\omega : (\forall t' \sqsubseteq t)\, t' \notin C\}$$

is a tree and $f = \mathrm{Exit}(T)$.

On the other hand, if $f = \mathrm{Exit}(T)$ for some tree $T \subseteq {}^{<\omega}\omega$, then the set

$$C := \{c \in {}^{<\omega}\omega : c \notin T \wedge (\forall t \sqsubseteq c)\, t \neq c \Rightarrow t \in T\}$$

is a 1-cloud and it represents f. □

Now, functions of the form $\mathrm{Exit}(T)$ where T is a leafless tree with only *one* branch are the simplest functions which are not continuous. Given $a \in {}^{\omega}\omega$, recall that $[[a]] \subseteq {}^{<\omega}\omega$ is the set of initial segments of a:

$$[[a]] := \{a \upharpoonright l : l \in \omega\}.$$

Hence,

$$\mathrm{Exit}([[a]])(x) = \begin{cases} 0 & \text{if } x = a, \\ \min\{l : x(l-1) \neq a(l-1)\} & \text{otherwise.} \end{cases}$$

That is, $\mathrm{Exit}([[a]])(x)$ is the level at which x deviates from a. Informally, $\mathrm{Exit}([[a]])$ is a discrete analogue of the function $f : \mathbb{R} \to \mathbb{R}$ defined as follows (for some $r \in \mathbb{R}$):

$$f(x) = \begin{cases} 0 & \text{if } x = r, \\ \frac{1}{x-r} & \text{otherwise.} \end{cases}$$

In the next section, we will see that all functions of this simple form cannot be everywhere dominated by fewer than 2^ω functions (of any complexity whatsoever). This is because a dominator of such a function must inherently contain the information of the single path.

5.2 Basic Construction (Vertical Coding)

We will now begin where the last section ended, and present the basic "vertical style" coding argument in its simplest form:

Proposition V.7. *Fix $a \in {}^\omega\omega$. If M is a transitive model of ZF such that some $g : ({}^\omega\omega)^M \to \omega$ in M satisfies*

$$(\forall x \in ({}^\omega\omega)^M)\, \mathrm{Exit}([[a]])(x) \leq g(x),$$

then $a \in M$.

Proof. Let M be any transitive model of ZF such that $a \notin M$. Consider any $g : ({}^\omega\omega)^M \to \omega$ in M. Suppose, towards a contradiction, that $(\forall x \in ({}^\omega\omega)^M)\, \mathrm{Exit}([[a]])(x) \leq g(x)$. Consider the following set:

$$B := \{t \in {}^{<\omega}\omega : g(x) \geq |t| \text{ for all } x \sqsupseteq t \text{ in } M\}.$$

Since B need not be a tree, let us define the tree T of those elements of B all of whose initial segments are also in B. Since $g \in M$, also $T \in M$. There cannot be any $x \in [T]$ in M, because if there was such an x, then we would have $g(x) \geq l$ for all $l \in \omega$, which contradicts the fact that g is well-defined. Hence, $(T \text{ is well-founded})^M$. Since being well-founded is absolute, T is well-founded.

On the other hand, $(\forall l \in \omega)\, a \restriction l \in B$. Let us explain. Fix $l \in \omega$. Any $x \in ({}^\omega\omega)^M$ that extends $a \restriction l$ differs from a (because $a \notin M$). Thus, x must first differ from a at some level $l' \geq l$, so $g(x) \geq \text{Exit}([[a]])(x) = l'$. Thus $(\forall l \in \omega)\, a \restriction l \in B$, and we have $(\forall l \in \omega)\, a \restriction l \in T$. Therefore $a \in [T]$, so T is not well-founded. □

The above proof is by contradiction, because Theorem VII.28 can only be reasonably proved by contradiction, and we want to show the difference between the arguments. This proposition implies that for each $a \in {}^\omega\omega$, if $g : {}^\omega\omega \to \omega$ satisfies $(\forall x \in ({}^\omega\omega)^{L[g]})\, \text{Exit}([[a]])(x) \leq g(x)$, then $a \in L[g]$. Certainly we have the following morphism (using notation which should be clear and which accompanies what we explained in Section 1.2):

$$
\begin{array}{ccc}
\text{Exit}([[a]]) & \leq' & g \\
\uparrow & \Downarrow & \downarrow \\
a & \in & L[g],
\end{array}
\tag{5.1}
$$

where we temporarily define $f \leq' g$ by $(\forall x \in ({}^\omega\omega)^{L[g]})\, f(x) \leq g(x)$.

A central aspect of the proposition above is that M need not include all of ${}^\omega\omega$. This contrasts with Theorem VII.28, where we really do need all reals available. That is, we expect it to be extremely difficult (if not impossible) to prove that for each $a \in {}^\omega\omega$, there is some Borel $f : {}^\omega\omega \to {}^\omega\omega$ such that if M is a transitive model of ZF containing some Borel $g : ({}^\omega\omega)^M \to {}^\omega\omega$ satisfying $(\forall x \in ({}^\omega\omega)^M)\, f(x) \leq^* g(x)$, then $a \in M$. Also, there is no burning need to generalize Theorem VII.28 in this way, whereas this generality of the proposition above leads to the important application to weak distributivity laws for complete Boolean algebras (Section 5.8).

Consider the bottom relation "$a \in L[g]$" of (5.1). If g is coded by some $c \in {}^{\omega}\omega$, then $a \in L[g]$ implies $a \in L[c]$. The relation "$a \in L[c]$" is called the *constructibility relation* between reals. Constructibility is a convenient relation because models of ZF have many closure properties and we may apply absoluteness arguments as done in the proposition above. Indeed, the results in this thesis were all discovered by treating constructibility as the essential relation, moving down to finer relations as a separate step.

Moving down to finer relations is needed to complete the overall picture. A deeper analysis of the proposition above allows us to strengthen the conclusion from simply $a \in M$ to a being explicitly definable in M by a formula. If we proceed as before and define T to be the set of elements of B all of whose initial segments are in B, then we will encounter a problem. Instead, what is relevant is the poset of elements of B ordered by extension. We dignify this generalization as a theorem, and it implies Theorem I.22 from the introduction. It is essentially the strongest coding theorem we can expect to prove where we encode real numbers into functions from ${}^{\omega}\omega$ to ω:

Theorem V.8. *Fix $a \in {}^{\omega}\omega$. If M is a transitive model of ZF such that some $g : ({}^{\omega}\omega)^M \to \omega$ in M satisfies*

$$(\forall x \in ({}^{\omega}\omega)^M)\, \mathrm{Exit}([[a]])(x) \leq g(x),$$

then a is Δ^1_1 definable in M using g as a predicate.

Proof. Fix M and g satisfying the hypothesis of the theorem. Define $B \subseteq {}^{<\omega}\omega$ in M exactly as in the proposition above. Note that B is defined (in M) by a Π^1_1 formula that uses g as a predicate. That is, B is Π^1_1 in g. We claim there is some $l \in \omega$

satisfying $(\forall l' \geq l)\, a \restriction l' \notin B$. If not, the poset of elements of B ordered by extension would be ill-founded, and therefore would be ill-founded in M, so there would exist $x \in ({}^{\omega}\omega)^M$ satisfying $(\exists^{\infty} l' \in \omega)\, g(x) \geq l'$, which is impossible. Now, fix such an l.

We claim that for each $l' \geq l$, $a(l')$ is the unique n satisfying $(a \restriction l')^\frown n \notin B$. Indeed, since $\mathrm{Exit}([[a]]) \leq g$, for each $l' \geq l$ we have

$$(\forall n \in \omega)\, a(l') \neq n \Rightarrow (a \restriction l')^\frown n \in B.$$

The other direction is given by the property we arranged l to have. Thus, we have the following definition (in M) for a:

$$a(l') = \begin{cases} a(l') & \text{if } l' < l, \\ n & \text{if } l' \geq l \text{ and } (\forall n' \neq n)(\forall x \sqsupseteq (a \restriction l')^\frown n' \text{ in } M)\, g(x) \geq l' + 1. \end{cases}$$

Since $\langle a(l') : l' < l \rangle$ can be coded by a single number, we have a Π^1_1 definition (in M) for a which uses g as a predicate. We also have a Σ^1_1 variant:

$$a(l') = \begin{cases} a(l') & \text{if } l' < l, \\ n & \text{if } l' \geq l \text{ and } (\exists x \sqsupseteq (a \restriction l')^\frown n \text{ in } M)\, g(x) < l' + 1. \end{cases}$$

Thus, a is Δ^1_1 definable in M using g as a predicate. □

Our picture is now complete, and we see four relations stacked on top of each other:

$$\begin{array}{ccc}
\mathrm{Exit}([[a]]) & \leq & g \\
\Big\uparrow & \Big\Downarrow & \Big\downarrow \\
\mathrm{Exit}([[a]]) & \leq' & g \\
\Big\uparrow & \Big\Downarrow & \Big\downarrow \\
a & \leq_{\Delta^1_1} & g \\
\Big\uparrow & \Big\Downarrow & \Big\downarrow \\
a & \in & L[g].
\end{array}$$

Now we may compute $\mathrm{cf}\, \mathcal{B}_\alpha(\omega, \leq)$ for all $\alpha \geq 1$:

Corollary V.9. *Fix $\alpha \geq 1$. We have*

$$\operatorname{cf}\mathcal{B}_\alpha(\omega, \leq) = 2^\omega.$$

Proof. By what we have said, there is certainly a morphism from $\mathcal{B}_\alpha(\omega, \leq)$ to $\langle {}^\omega\omega, \leq_{\Delta^1_1}\rangle$. The cofinality of $\langle {}^\omega\omega, \leq_{\Delta^1_1}\rangle$ is 2^ω, because each real has only countably many reals Δ^1_1 reducible to it, and we are done. □

It goes without saying that the arguments of this section carry over to functions with domain ${}^\omega 2$ instead of ${}^\omega\omega$. The encoding $a \mapsto \mathrm{Exit}([[a]])$ we informally call *vertical coding*, because the information inherent within a is laid out vertically in the tree ${}^{<\omega}\omega$. We will present a different encoding scheme in Section 5.4: *horizontal coding*. As we will see, neither method is strictly better than the other, and some situations require us to use one but not the other.

5.3 Blow-Up Trees

The purpose of this section is to analyze exactly how sloppy we can be with our encoding scheme $a \mapsto f$ so that still $a \in L[g]$ whenever $f \leq g$. The reader may skip to the next section with no loss of continuity. We saw that the scheme $a \mapsto \mathrm{Exit}([[a]])$ worked, but we used the conspicuously defined set

$$B = \{t \in {}^{<\omega}\omega : g(x) \geq |t| \text{ for all } x \sqsupseteq t\}$$

in our argument. We shall see that indeed we can be quite sloppy, and our observations may be of use to an analyst.

To begin, let us temporarily think of elements of ${}^\omega\omega$ as simply points in a space rather than paths through a tree, and describe properties of functions from this point of view. Recall the notation $f``(U) := \{f(x) : x \in U\}$. What we say applies to functions from an arbitrary uncountable Polish space X to $\mathbb{R}$, but let us stick to functions from ${}^\omega\omega$ to ω to keep our discussion focused.

Definition V.10. $a \in {}^\omega\omega$ is a *blow-up* point of $f : {}^\omega\omega \to \omega$ if $f``(U)$ is unbounded for each neighborhood U of a. We say that a is a *pure blow-up* point of f if for each $n \in \omega$, there is some neighborhood U of a such that for all $x \in U - \{a\}$, $f(x) \geq n$.

That is, a is a blow-up point of $f : {}^\omega\omega \to \omega$ iff $\limsup_{x\to a} f(x) = \omega$ and a is a pure blow-up point iff $\lim_{x\to a} f(x) = \omega$. Recall that given $t \in {}^{<\omega}\omega$, $[t]$ is the set of elements of ${}^\omega\omega$ which extend t. When we investigated continuous functions, blow-up points did not appear:

Proposition V.11. *$f : {}^\omega\omega \to \omega$ is dominated by a continuous function iff f has no blow-up points.*

Proof. If $f : {}^\omega\omega \to \omega$ is dominated by a continuous function $g : {}^\omega\omega \to \omega$, then given any $x \in {}^\omega\omega$, there is some neighborhood U of x such that g is constant on U, so x cannot be a blow-up point of f.

On the other hand, suppose f has no blow-up points. For each $x \in {}^\omega\omega$, there is some shortest finite initial segment s_x of x such that $f``([s_x])$ is bounded. Let $g : {}^\omega\omega \to \omega$ be the function

$$g(x) := \max f``([s_x]).$$

Since $(\forall x \in {}^\omega\omega)\, f(x) \in f``([s_x])$, we have $f \leq g$. Furthermore, one can check that the sets $[s_x]$ form a partition of ${}^\omega\omega$, so g is continuous. $\square$

We first encounter blow-up points when looking at $\mathcal{F}_1$ functions represented by 1-clouds. Recall that by Proposition V.6, functions represented by 1-clouds are precisely those functions of the form $\mathrm{Exit}(T)$ for some tree $T \subseteq {}^{<\omega}\omega$.

Proposition V.12. *$f : {}^{\omega}\omega \to \omega$ is dominated by a function represented by a 1-cloud iff $f(x) = 0$ for each blow-up point x of f.*

Proof. Suppose $f : {}^{\omega}\omega \to \omega$ is dominated by an $\mathcal{F}_1$ function $g : {}^{\omega}\omega \to \omega$ represented by a 1-cloud. By the definition of a 1-cloud, $g(x) = 0$ for each blow-up point x of g. Since g dominates f, every blow-up point of f is a blow-up point of g. Hence, $f(x) = 0$ for each blow-up point of f.

For the other direction, suppose $f : {}^{\omega}\omega \to \omega$ is such that $f(x) = 0$ for each blow-up point x of f. Let $C_g := \{t \in {}^{<\omega}\omega : f``([t])$ is bounded but $f``([t'])$ is unbounded for every proper initial segment t' of $t\}$. Notice that C_g is a 1-cloud. Let $\tilde{g} : C_g \to \omega$ be defined by

$$\tilde{g}(t) := \max f``([t]).$$

Let $g : {}^{\omega}\omega \to \omega$ be the function specified by C_g and $\tilde{g} : C_g \to \omega$ as in Proposition V.3. That is, $g(x) = \max\{\tilde{g}(x \upharpoonright l) : x \upharpoonright l \in C_g\}$. By that proposition, g is $\mathcal{F}_1$, and by applying Proposition V.5 to the 1-cloud C_g and function $\tilde{g}$, we get a 1-cloud C satisfying $g \leq \mathrm{Rep}(C)$. $\square$

Now that we have characterized which functions are everywhere dominated by either continuous or Baire class one functions, let us return to our discussion of encoding reals into functions. One might make the mistake of thinking the only crucial part of Proposition V.7 was that the function Exit([[a]]) had a blow-up point (the point a) not in the ground model. The following simple observation shows that more is needed:

Counterexample V.13. *Let M be a transitive model of ZF. There is a Borel function $f : {}^{\omega}\omega \to \omega$ such that $f \upharpoonright M \in M$, and yet for each $a \in {}^{\omega}\omega$ (including those a not in M) and each neighborhood U of a, $f``(U \cap M)$ is unbounded.*

Proof. Let $f : {}^\omega\omega \to \omega$ be defined by $f(x) := 0$ if $(\exists^\infty n)\, x(n) \neq 0$, and $f(x) := n$ if n is the first number such that $x(m) = 0$ for all $m \geq n$. Certainly, f is Borel and $f \restriction M \in M$. Let S be the set of all $x \in {}^\omega\omega$ satisfying $(\forall^\infty n)\, x(n) = 0$. We have that $S \subseteq M$. Given any $y \in {}^\omega\omega$ and any neighborhood U of y, $f``(U \cap S)$ is unbounded, and so $f``(U \cap M)$ is unbounded. $\square$

The fact that a is a *pure* blow-up point of Exit([[a]]) in Proposition V.7 is the crucial point. To push the argument to work with a more general function f, we need to replace the set B within the proof with the more technical poset $\langle W, \prec \rangle$:

Proposition V.14. *Let M be a transitive model of ZF. Let $f : {}^\omega\omega \to \omega$ and $a \in {}^\omega\omega$ be such that for each $n \in \omega$, there is some neighborhood U of a satisfying*

$$(\forall x \in U \cap M - \{a\})\, n \leq f(x)$$

(which happens when a is a pure blow-up point of f). Let $g : ({}^\omega\omega)^M \to \omega$ in M satisfy

$$(\forall x \in ({}^\omega\omega)^M)\, f(x) \leq g(x).$$

Then $a \in M$.

Proof. For each $n \in \omega$, let

$$S_n := \{t \in {}^{<\omega}\omega : g(x) \geq n \text{ for all } x \sqsupseteq t \text{ in } M\}.$$

Notice that each ${}^{<\omega}\omega - S_n$ is a tree. Let $\langle W, \prec \rangle$ be the poset

$$W := \{\langle t_0, ..., t_n \rangle : n \in \omega \wedge t_0 \in S_0 \wedge ... \wedge t_n \in S_n \wedge t_0 \sqsubseteq ... \sqsubseteq t_n\},$$

where $w_2 \prec w_1$ iff w_1 is a proper initial segment of w_2. First, note that $W \in M$. This is because $g \in M$, and therefore $\langle S_n : n \in \omega \rangle \in M$. Next,

$$(\langle W, \prec \rangle \text{ is well-founded})^M.$$

This is because if there was some infinite decreasing sequence through W in M, then there would exist $x \in ({}^{\omega}\omega)^M$ as well as $\langle t_n \in S_n : n \in \omega \rangle \in M$ satisfying $t_n \sqsubseteq x$ for all $n \in \omega$. This would imply that $g(x) \geq n$ for all $n \in \omega$, which is impossible.

Since $\langle W, \prec \rangle$ is well-founded in M and being well-founded is absolute, W is indeed well-founded. Now, assume towards a contradiction that $a \notin M$. Suppose we are given $\langle t_0, ..., t_n \rangle \in W$ satisfying $t_0 \sqsubseteq ... \sqsubseteq t_n \sqsubseteq a$. By hypothesis and since $a \notin M$, there is some neighborhood U of a such that $(\forall x \in U \cap M)\, n \leq f(x)$. Pick $t_{n+1} \sqsubseteq a$ so that $t_n \sqsubseteq t_{n+1}$ and $[t_{n+1}] \subseteq U$. Now, for any $x \in [t_{n+1}] \cap M$, $n \leq f(x) \leq g(x)$. Thus, $t_{n+1} \in S_{n+1}$. Hence, $\langle t_0, ..., t_{n+1} \rangle \in W$ and $\langle t_0, ..., t_{n+1} \rangle \prec \langle t_0, ..., t_n \rangle$. By applying this construction inductively starting with $\emptyset$, we obtain an infinite decreasing sequence through $\langle W, \prec \rangle$. This contradicts $\langle W, \prec \rangle$ being well-founded, and the proof is complete. □

The proof above illustrates a common idea used in descriptive set theory. Namely, $\langle W, \prec \rangle$ is a *tree of attempts to build something which does not exist.* This tree was hidden in our previous arguments because it was obscured by a more prominent tree: ${}^{<\omega}\omega$. Now, $\langle W, \prec \rangle$ has two essential properties. First, it cannot have any branches (infinite decreasing sequences), because given a branch there must exist a point x satisfying $g(x) \geq n$ for all n, which is impossible. Second, the way f is defined makes it so if g dominates f, then $\langle W, \prec \rangle$ contains many nodes. We might want to modify the definition of $\langle W, \prec \rangle$ to handle other functions f, but we need to make sure the first property is still satisfied. The following definition accomplishes this:

Definition V.15. Let X be a set and $g : X \to \omega$ be a function. A poset $\langle W, \prec \rangle$ is a *blow-up tree* for g if the following conditions are satisfied:

1) each element of W is a finite decreasing sequence $\langle C_0, ..., C_n \rangle$ of subsets of X

where for each k satisfying $0 \leq k \leq n$,

$$(\forall x \in C_k)\, g(x) \geq k;$$

2) W is closed under initial segments;

3) if $w_1, w_2 \in W$, then $w_2 \prec w_1$ iff w_1 is a proper initial segment of w_2;

4) If $\langle C_0 \rangle \succ \langle C_0, C_1 \rangle \succ ...$ is an infinite decreasing sequence of elements of W, then $\bigcap_{n \in \omega} C_n \neq \emptyset$.

By conditions 1) and 4), a blow-up tree is necessarily well-founded. For demonstration purposes, we will repeat the proof of the proposition above but for $\mathbb{R}$ instead of ${}^{\omega}\omega$ and with a slightly weaker hypothesis:

Proposition V.16. *Let M be a transitive model of ZF. Let $f : \mathbb{R} \to \omega$ be a function and let $a \in \mathbb{R}$ be a point such that for each $n \in \omega$, there is some closed set C containing a with a Borel code in M satisfying*

$$(\forall x \in C \cap M - \{a\})\, n \leq f(x).$$

Suppose there is some $g : \mathbb{R}^M \to \omega$ in M satisfying

$$(\forall x \in \mathbb{R}^M)\, f(x) \leq g(x).$$

Then $a \in M$.

Proof. Assume, towards a contradiction, that there is such a g but $a \notin M$. Within M, we will define a blow-up tree $\langle W, \prec \rangle$ for g. Let W be the set of all finite decreasing sequences $\langle C_0, ..., C_n \rangle$ of compact subsets of $\mathbb{R}^M$ such that for each $k \in [0, n]$ and $x \in C_k$, $g(x) \geq k$. Conditions 1) to 3) of Definition V.15 are satisfied automatically. Since the intersection of an infinite decreasing sequence of compact sets is nonempty, 4) is satisfied, so $\langle W, \prec \rangle$ is indeed a blow-up tree for g.

Since it is a blow-up tree, it is well-founded in M. Since being well-founded is absolute, it is indeed well-founded. On the other hand, it is not difficult to argue from the hypothesis of the theorem that $(W, \prec)$ must have an infinite decreasing sequence, and so is not well-founded. This is a contradiction. $\square$

As a corollary, we have a result of potential interest to an analyst. Our choice of $(x-a)^{-1}$ as an example is arbitrary:

Corollary V.17. *Fix $a \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be the function*

$$f(x) := \begin{cases} 0 & \text{if } x = a, \\ \dfrac{1}{x-a} & \text{otherwise.} \end{cases}$$

If $g : \mathbb{R} \to \mathbb{R}$ is a function which everywhere dominates f, then $a \in L[g]$. Hence, if g is also Borel, then $a \in L[c]$ where c is any Borel code for g.

Proof. If g is Borel and c is a code for g, then $L[g] \subseteq L[c]$. Thus, it suffices to prove the first claim. Define the function $\tilde{f} : \mathbb{R} \to \omega$ by

$$\tilde{f}(x) := \lfloor f(x) \rfloor.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be any function which everywhere dominates $\tilde{f}$. Note that $g \cap L[g] \in L[g]$. Being a transitive model of ZF, $L[g]$ contains all rational numbers, and therefore contains all Borel codes for closed intervals with rational endpoints. Note that for each $n \in \omega$, there are rational numbers $r_1, r_2 \in \mathbb{Q}$ satisfying $a \in [r_1, r_2]$ and

$$(\forall x \in [r_1, r_2] - \{a\})\, n \leq \tilde{f}(x).$$

We may now apply the proposition above with $L[g]$, $\tilde{f}$, and $g \cap L[g]$ to conclude that $a \in L[g]$. $\square$

Let us remark that there is a limitation to how sloppy we can be in creating a function all of whose dominators construct $a \in {}^\omega\omega$. Specifically, suppose $a \in {}^\omega\omega$ and $f : {}^\omega\omega \to \omega$ are such that for each $n \in \omega$ and each neighborhood U of a, there is some open (and non-empty) $U_n \subseteq U$ satisfying $(\forall x \in U_n)\, f(x) \geq n$. It does not follow that if $g : {}^\omega\omega \to \omega$ satisfies $f \leq g$, then a is definable from g in any sense.

5.4 Modifying the Encoding (Horizontal Coding)

In Section 5.2, we saw how a real $a \in {}^\omega\omega$ is encoded into the function $\mathrm{Exit}([[a]])$ (in what may be described as a *vertical* way). For technical reasons, which will be clear in Section 5.6, we need an alternate coding scheme. Let X be a set and $A \subseteq X$. Let $f_A : {}^\omega X \to \omega$ be the function

$$f_A(x) := \begin{cases} 0 & \text{if } (\forall l \in \omega)\, x(l) \notin A, \\ l+1 & \text{if } x(l) \in A \text{ and } (\forall l' < l)\, x(l') \notin A. \end{cases}$$

Note that $f_A = \mathrm{Exit}(T)$ where T is the tree of all $t \in {}^{<\omega}\omega$ satisfying

$$(\forall l' \in \mathrm{Dom}(t))\, t(l') \notin A.$$

For each $t \in T$,

$$A = \{z \in X : t^\frown z \notin T\}.$$

This justifies calling the encoding scheme $A \mapsto f_A$ *horizontal* coding, because the information within A is laid out horizontally in the tree ${}^{<\omega}X$. Equivalently, $f_A = \mathrm{Rep}(C)$ where C is the cloud of all $t \in {}^{<\omega}\omega$ satisfying $t(|t|-1) \in A$ but $(\forall l' < |t|-1)\, t(l') \notin A$. Thus, when $X = \omega$, f_A is represented by a 1-cloud, and is therefore $\mathcal{F}_1$, and hence is Baire class one.

When $X = \omega$, we have an analogue of Theorem V.8 but with a different proof. However, when $X = \mathbb{R}$, we get an encoding scheme for subsets of $\mathbb{R}$ rather than elements of ${}^{\omega}\omega$, which is beyond the scope of vertical coding. The point is that while there are only $|X|^{\omega}$ paths through the tree ${}^{<\omega}X$, there are $2^{|X|}$ subsets of X. Very informally, we may say "there is more room to store information horizontally".

Proposition V.18. *Fix a set X. Fix $A \subseteq X$. Let $f_A : {}^{\omega}X \to \omega$ be defined as above. Let M be a transitive model of ZF with $X \in M$ and containing some $g : ({}^{\omega}X)^M \to \omega$ satisfying*

$$(\forall x \in ({}^{\omega}X)^M)\, f_A(x) \leq g(x).$$

Then $A \in M$. Moreover, there is some $t \in {}^{<\omega}X$ satisfying

$$A = \{z \in X : g(x) \geq |t| + 1 \textit{ for all } x \sqsupseteq t^\frown z \textit{ in } M\}.$$

Proof. It suffices to show the second claim. As in our previous arguments, define

$$B := \{t \in {}^{<\omega}X : g(x) \geq |t| \text{ for all } x \sqsupseteq t \text{ in } M\}.$$

We must find a $t \in {}^{<\omega}X$ satisfying

$$A = \{z \in X : t^\frown z \in B\},$$

and we will be done. By the hypothesis on g and the definition of f_A, for each $z \in X$, $z \in A$ implies $\langle z \rangle \in B$. If conversely for each $z \in X$, $\langle z \rangle \in B$ implies $z \in A$, then we have

$$A = \{z \in X : \langle z \rangle \in B\},$$

and we are done by defining $t := \emptyset$. If not, then fix some $x_0 \in X$ satisfying $\langle x_0 \rangle \in B$ but $x_0 \notin A$.

Again by the hypothesis on g and the definition of f_A, for each $z \in X$, $z \in A$ implies $\langle x_0, z\rangle \in B$. Here it is important that $x_0 \notin A$. Again, if the converse holds that $\langle x_0, z\rangle \in B$ implies $z \in A$, then

$$A = \{z \in X : \langle x_0, z\rangle \in B\},$$

and we are done by defining $t := \langle x_0\rangle$. If not, we may fix $x_1 \in X$ satisfying $\langle x_0, x_1\rangle \in B$ but $x_1 \in A$. We may continue like this, but we claim that the procedure terminates in a finite number of steps.

Assume, towards a contradiction, that it does not terminate. The sequence

$$x := \langle x_0, x_1, ...\rangle$$

we have constructed has all its initial segments in B. However, x need not be in M. We handle this situation as before: let T be the set of all elements of B all of whose initial segments are also in B. The tree T is ill-founded because x is a path through it. Since being ill-founded is absolute, T has some path x' in M. We now have $(\forall l \in \omega)\, g(x') \geq l$, which is impossible. $\square$

In some sense, the proof of Proposition V.18 is more aesthetically pleasing than that of Theorem V.8; the definition of A within the transitive model M has a particularly simple form.

Corollary V.19. *For each $A \subseteq {}^{\omega}\omega$, there is a function $f : {}^{\omega}\omega \to \omega$ such that whenever $g : {}^{\omega}\omega \to \omega$ is any function which satisfies $f \leq g$, then A is $\mathbf{\Delta}^1_1$ in g.*

Proof. Let $X := {}^{\omega}\omega \sqcup {}^{\omega}\omega$. Let $A' \subseteq X$ be such that its intersection with the first ${}^{\omega}\omega$ is A, and its intersection with the second ${}^{\omega}\omega$ is ${}^{\omega}\omega - A$. Fix a (canonical) bijection η between ${}^{\omega}\omega$ and ${}^{\omega}X$. Define $f : {}^{\omega}\omega \to \omega$ to be the function $f(x) = f_{A'}(\eta(x))$. Now suppose g satisfies $f \leq g$. Let $g' : {}^{\omega}X \to \omega$ be the function $g'(x) = g(\eta^{-1}(x))$. We have $f_{A'} \le g'$. By the proposition above, we see that A' is $\mathbf{\Pi}^1_1$ in g' (we require the boldface version of the pointclass because the $t \in {}^{<\omega}X$ given by the proposition is coded by a real). By our provision that A' is the disjoint union of A and ${}^{\omega}\omega - A$, we see that in fact A is $\mathbf{\Delta}^1_1$ in g'. Since the bijection η is canonical, we have that A is $\mathbf{\Delta}^1_1$ in g. □

Of course, this corollary also holds for functions from any Polish space to ω. We easily get the following:

Corollary V.20. *For each $A \subseteq {}^{\omega}\omega$, there is a function $f : {}^{\omega}\omega \to \omega$ such that whenever $g : {}^{\omega}\omega \to \omega$ is any function which satisfies $f \le g$, then $A \in L({}^{\omega}\omega, A)$.*

Also from Corollary V.19 we get an alternate way to compute the cofinality of the set of all functions from ${}^{\omega}\omega$ to ω ordered pointwise:

Corollary V.21. $\operatorname{cf} \operatorname{All}(\omega, \le) = 2^{2^{\omega}}$.

Finally, let us remark that the proof of Proposition V.18 has a simple visualization when we think of elements of ${}^{\omega}X$ as points in a space rather than paths through a tree. That is, given $A \subseteq X$, we may think of ${}^{\omega}X$ as being partitioned into $|X|$ *blocks* of the form $[\langle z \rangle]$ for $z \in X$. The function f_A assigns 1 to each point in a block corresponding to an element of A. Now suppose $f_A \le g$. For each block which f assigns 1 to each point within, g must assign at least 1 to each point within. However, assuming g exists in a model which does not contain A, the function g is going to make a mistake and assign at least 1 to each point in a block $[\langle x_0 \rangle]$ which does *not* correspond to an element of A. Indeed, g is overzealous. If we focus on that block, we may repeat the argument. That is, that block is partitioned into $|X|$ smaller blocks of the form $[\langle x_0, z \rangle]$ for $z \in X$. The function f_A assigns 2 to each smaller block corresponding to an element of A. Since $f_A \le g$ but "g does not know about

A", g will be overzealous and assign at least 2 to each point in a block which does not correspond to an element of A, etc.

5.5 Morphisms Involving Trees and Clouds

Well-founded trees were fundamental for computing $\mathrm{cf}\,\mathcal{B}_0(\omega, \le)$ and clouds were fundamental for computing $\mathrm{cf}\,\mathcal{B}_\alpha(\omega, \le)$ for $\alpha \ge 1$. In this section, we will complete the picture by relating the inclusion ordering on well-founded trees to the inclusion ordering on clouds. We hope to convince the reader that the combinatorics of well-founded trees and clouds is the heart of the situation, and extra complexity arises when relating these structures to functions from ${}^\omega\omega$ to ω. Some of what we say extends to subsets of ${}^{<\kappa}\kappa$, where we have already defined what it means to be a cloud in this context, and the property that a tree is well-founded tree is replaced with the property of not having any length κ branches. We have faith that the reader can carry out such generalizations without trouble. However, there is subtlety because both the property of $S \subseteq {}^{<\kappa}\kappa$ not having any length κ branch and the property of being a cloud are not in general absolute between models of set theory when $\kappa > \omega$.

Recall that $\mathcal{W}$ is the set of well-founded subtrees of ${}^{<\omega}\omega$. For the sake of this section, let us introduce a corresponding notation for clouds:

Definition V.22. $\mathcal{C}$ is the collection of subsets of ${}^{<\omega}\omega$ which are clouds.

Given a cloud $C \in \mathcal{C}$, recall that $\mathrm{Rep}(C) : {}^\omega\omega \to \omega$ is a Baire class one function. Note that for $C_1, C_2 \in \mathcal{C}$,

$$C_1 \subseteq C_2 \Rightarrow \mathrm{Rep}(C_1) \le \mathrm{Rep}(C_2).$$

Given a function from ${}^\omega\omega$ to ω, there is a cloud we may extract from it. Namely, the set B that we have been using in our arguments in this chapter:

Definition V.23. Given a function $g : {}^{\omega}\omega \to \omega$, $\mathrm{Cloud}(g) \in \mathcal{C}$ is defined by

$$\mathrm{Cloud}(g) := \{t \in {}^{<\omega}\omega : g(x) \geq |t| \text{ for all } x \sqsupseteq t\}.$$

The inclusion ordering on clouds reduces to the everywhere domination ordering of functions from ${}^{\omega}\omega$ to ω. That is, we see that given $C \in \mathcal{C}$ and $g : {}^{\omega}\omega \to \omega$,

$$\mathrm{Rep}(C) \leq g \Rightarrow C \subseteq \mathrm{Cloud}(g).$$

That shows that if Γ is any pointclass of functions from ${}^{\omega}\omega$ to ω which includes all Baire class one functions, there is a morphism from $\langle \Gamma, \leq \rangle$ to $\langle \mathcal{C}, \subseteq \rangle$. We get a morphism in the other direction when we restrict to only Baire class one functions from ${}^{\omega}\omega$ to ω. Indeed, by Proposition V.5, each function in $\mathcal{B}_1(\omega, \leq)$ is below one represented by a cloud. Thus, if $\phi_- : \mathcal{B}_1(\omega) \to \mathcal{C}$ is a map which selects such a cloud and $\phi_+ = \mathrm{Rep}$, then $\langle \phi_-, \phi_+ \rangle$ is a morphism from $\langle \mathcal{C}, \subseteq \rangle$ to $\mathcal{B}_1(\omega, \leq)$. Thus, there are morphisms in both directions between $\langle \mathcal{C}, \subseteq \rangle$ and $\mathcal{B}_1(\omega, \leq)$.

As a consequence of Theorem V.8, there is a morphism from $\mathcal{B}_1(\omega, \leq)$ to $\langle \mathcal{P}(\omega), \leq_{\Delta^1_1} \rangle$. The reason for Δ^1_1 is because the definition of B within the proof of that theorem involves a real quantifier. The quantification is absorbed into the definition of $\mathrm{Cloud}(g)$. When we restrict attention to $\langle \mathcal{C}, \subseteq \rangle$, we see a sharper form of reducibility (we use Turing reducibility $\leq_T$ as an example):

Proposition V.24. *For each $A \subseteq \omega$, there is some $C_A \in \mathcal{C}$ such that whenever $C \in \mathcal{C}$ satisfies $C_A \subseteq C$, there exists some $t \in {}^{<\omega}\omega$ satisfying*

$$A = \{n \in \omega : t^\frown n \in C\}.$$

Hence, there is a morphism from $\langle \mathcal{C}, \subseteq \rangle$ to $\langle \mathcal{P}(\omega), \leq_T \rangle$.

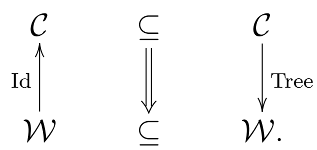
Proof. Let C_A be the set of all $t \in {}^{<\omega}\omega$ satisfying $t(|t|-1) \in A$ but $(\forall l' < |t|-1)\, t(l') \notin A$. This is precisely the cloud we described at the beginning of Section 5.4 satisfying

$f_A = \mathrm{Rep}(C_A)$. Now let C be any cloud satisfying $C_A \subseteq C$. First, note that for all $n \in \omega$, $n \in A \Rightarrow \emptyset^\frown n \in C_A$. If the converse holds for all n, then we are done by defining $t := \emptyset$. Otherwise, we may fix $x_0 \in \omega$ with $x_0 \notin A$ but $\langle x_0 \rangle \in C$. Now, for all $n \in \omega$, $n \in A \Rightarrow \langle x_0 \rangle^\frown n \in C_A$. Again, either the converse implication holds for all n and we are done, or we may continue by fixing an $x_1 \in \omega$ satisfying $x_1 \notin A$ but $\langle x_0, x_1 \rangle \in C$. The procedure must eventually terminate, because otherwise we would have a path which hits C at infinitely many places, contradicting C being a cloud. □

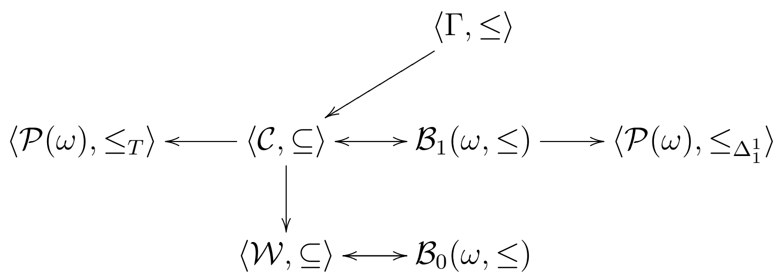
To connect well-founded trees to clouds, we have the following:

Definition V.25. Given $C \in \mathcal{C}$, the tree $\mathrm{Tree}(C) \in \mathcal{W}$ is the set of elements of C all of whose initial segments are also in C.

We now see a morphism from $\langle \mathcal{C}, \subseteq \rangle$ to $\langle \mathcal{W}, \subseteq \rangle$:



At this point, we have a detailed picture of how well-founded trees and clouds fit into our investigation. Let Γ be any pointclass of functions from ${}^\omega\omega$ to ω which includes all Baire class one functions. In the following diagram, an arrow represents the existence of a morphism, and a double arrow represents the existence of a morphism in each direction:



The morphisms between the bottom two relations are given by Propositions IV.6 and IV.7. This diagram shows that there must exist a morphism from $\mathcal{B}_1(\omega, \leq)$ to $\mathcal{B}_0(\omega, \leq)$, which we should not expect a priori. A similar surprise is a morphism from $\langle \Gamma, \leq \rangle$ to $\mathcal{B}_1(\omega, \leq)$.

Finally, although we have been discussing clouds which are subsets of ${}^{<\omega}\omega$, we could just have well considered clouds which are subsets of ${}^{<\omega}2$. We leave it as an exercise to the reader to show that there is a morphism in each direction between these two collections of clouds ordered by inclusion.

5.6 Main Coding Theorems

We change gears slightly from descriptive set theory to combinatorial set theory, although the core ideas are the same. The arguments we have given, using vertical and horizontal coding, generalize easily (modulo a few fascinating technicalities) to handle functions from ${}^{\kappa}\lambda$ to κ for infinite κ and λ. We insist that the functions have domain ${}^{\kappa}\lambda$ instead of λ^{κ}, because the transitive model involved needs to *understand the structure of the domain of the functions.* An arbitrary transitive model M which contains the ordinal λ^{κ} need not think there is a bijection between that ordinal and $({}^{\kappa}\lambda)^M$. We believe that considering functions from ${}^{\kappa}\lambda$ to κ is the fundamental way to understand the situation. These coding theorems will have significant applications at the end of the chapter, where we will use them to get new implications between distributivity laws for complete Boolean algebras.

Throughout this section, for each $A \subseteq \lambda$, let $f_A : {}^{\kappa}\lambda \to \kappa$ be the function

$$f_A(x) := \begin{cases} 0 & \text{if } (\forall \alpha < \kappa)\, x(\alpha) \notin A, \\ \alpha + 1 & \text{if } x(\alpha) \in A \text{ and } (\forall \beta < \alpha)\, x(\beta) \notin A. \end{cases}$$

We may call f_A the *horrizontal encoding* of A.

Proposition V.26. *For each $A \subseteq \lambda$, whenever M is a transitive model of ZF with ${}^{\kappa}\lambda \in M$ and some $g : {}^{\kappa}\lambda \to \kappa$ in M satisfies $f_A \leq g$, then $A \in M$.*

Proof. Define the set

$$B := \{t \in {}^{<\kappa}\lambda : g(x) \geq \mathrm{Dom}(t) \text{ for all } x \sqsupseteq t \text{ in } M\}.$$

We may argue, just as in Proposition V.18, that there is some $t \in {}^{<\kappa}\lambda$ satisfying

$$A = \{z \in X : t^\frown z \in B\}.$$

That is, we start defining a sequence $x = \langle x_0, x_1, ...\rangle$ such that each $x_\alpha \notin A$ and $x \restriction (\alpha + 1) \in B$. At limit stages, we take the sequence to be the limit of what we have constructed so far. If the procedure does not terminate at a stage before κ (to produce the desired t), then we have an $x \in {}^{\kappa}\lambda$ (which by hypothesis is in M) satisfying $(\forall \alpha < \kappa)\, x \restriction \alpha \in B$. Hence, $(\forall \alpha < \kappa)\, g(x) \geq \alpha$, which is impossible. □

For important reasons (the applications to weak distributivity laws for complete Boolean algebras), we need to weaken the hypothesis that ${}^{\kappa}\lambda \in M$. We have already seen one way of doing this, whose statement we repeat now to compare with the proposition above and those which will follow:

Proposition V.27. *For each $A \subseteq \lambda$, whenever M is a transitive model of ZF with $\lambda \in M$ and some $g : {}^{\omega}\lambda \to \omega$ in M satisfies*

$$(\forall x \in ({}^{\omega}\lambda)^M)\, f_A(x) \leq g(x),$$

then $A \in M$.

Proof. This is simply the proof of Proposition V.18 with $X = \lambda$. □

Note that we can replace the hypothesis that $\lambda \in M$ with the hypothesis that $\lambda = M \cap \mathrm{Ord}$ and the graph of g is adjoined to M as a predicate. Then if $(\forall x \in ({}^{\omega}\lambda)^M)\, f_A(x) \leq g(x)$, then A is a definable class within M (using g as a predicate).

The way Proposition V.27 handles the technicality that ${}^{\omega}\lambda$ need not be a subset of M is by using the absoluteness of trees being well-founded. However, this only applies to the case when $\kappa = \omega$, because for $\kappa > \omega$ it is *not* absolute between models of ZFC whether subtrees of ${}^{<\kappa}\lambda$ have length κ branches. Indeed, if M is a model of ZFC and $T \in M$ is such that $(T \text{ is a Suslin tree})^M$, then if V is a forcing extension of M by T, there will be a length ω_1 branch through T in V (but of course not in M). This proves that we need some additional assumption for getting the absoluteness of the existence of a length ω_1 branch through a subtree of ${}^{<\omega_1}\lambda$. One may ask if there is perhaps a completely different way to prove the analogue of Proposition V.27 where we replace ω with ω_1. Again, Suslin trees tell us the answer is no:

Counterexample V.28. *The following is* ***not*** *a theorem of ZFC (for any λ): for each $A \subseteq \omega_1$, there is a function $f : {}^{\omega_1}\lambda \to \omega_1$ such that whenever M is a transitive model of ZFC with $\lambda \in M$ and ${}^{<\omega_1}\lambda \subseteq M$, and some $g : ({}^{\omega_1}\lambda)^M \to \omega_1$ in M satisfies*

$$(\forall x \in ({}^{\omega_1}\lambda)^M)\, f(x) \leq g(x),$$

then $A \in M$.

Proof. Let M be a transitive model of ZFC which contains a (pruned) Suslin tree $T \subseteq {}^{<\omega_1}2$. Assume V is a forcing extension of M by T. Since M and V have the same ordinals, $\lambda \in M$. It is well-known that Suslin trees are (ω, ∞)-distributive, so all countable sequences in V of elements from M are already in M. In particular, ${}^{<\omega_1}\lambda \subseteq M$. Now (within M), the forcing is ω_1-c.c. Hence (within M), by Corollary II.37 the forcing is weakly $(\lambda^{\omega_1}, \omega_1)$-distributive. Thus, for each $f : {}^{\omega_1}\lambda \to \omega_1$ there is some

$g : ({}^{\omega_1}\lambda)^M \to \omega_1$ in M satisfying $f(x) \le g(x)$ for all $x \in ({}^{\omega_1}\lambda)^M$. On the other hand, $A \notin M$ where A codes the generic path through the tree T. $\square$

One way to get the desired absoluteness of the existence of length ω_1 branches through trees of height ω_1 is to assume the *tower number* $\mathfrak{t}$ is $> \omega_1$. Recall that $\mathfrak{t}$ is the smallest length of a sequence

$$\langle A_\alpha \in [\omega]^\omega : \alpha < \kappa \rangle$$

of infinite subsets of ω satisfying $(\forall \alpha < \beta < \kappa)\, A_\alpha \supseteq^* A_\beta$ but there is no $A \in [\omega]^\omega$ satisfying $(\forall \alpha < \kappa)\, A_\alpha \supseteq^* A$ (where $A \subseteq^* B$ means $A - B$ is finite). It is not hard to see that $\omega_1 \le \mathfrak{t} \le 2^\omega$. See [2] for more on $\mathfrak{t}$ and related cardinals. The absoluteness trick in this next proposition is burrowed from Farah in [15], who got the idea from Dordal in [10], who got the idea from Booth.

Proposition V.29. *Assume $\omega_1 < \mathfrak{t}$. For each $A \subseteq \omega_1$, whenever M is a transitive model of ZF with $\omega_1 \in M$ and $\mathcal{P}(\omega) \subseteq M$ and some $g : ({}^{\omega_1}\omega_1)^M \to \omega_1$ in M satisfies*

$$(\forall x \in ({}^{\omega_1}\omega_1)^M)\, f_A(x) \le g(x),$$

then $A \in M$.

Proof. Note that $\mathcal{P}(\omega) \subseteq M$ implies ${}^{<\omega_1}\omega_1 \subseteq M$, but we will use the assumption $\mathcal{P}(\omega) \subseteq M$ for an additional purpose. Define $B \subseteq {}^{<\omega_1}\omega_1$ just as in Proposition V.26. Assume, towards a contradiction, that $A \notin M$. As we argued in Proposition V.26, there is an $x \in {}^{\omega_1}\omega_1$ (in V) satisfying $(\forall \alpha < \omega_1)\, x \restriction \alpha \in B$. It is important that ${}^{<\omega_1}\omega_1 \subseteq M$, because otherwise we might get stuck at some stage strictly before ω_1. We claim that in fact $x \in M$. Once we show this, we will have our contradiction.

To prove the claim, let $F : {}^{<\omega_1}\omega_1 \to [\omega]^\omega$ be a function in M such that for all $t_1, t_2 \in {}^{<\omega_1}\omega_1$, the following hold:

1) $t_2 \sqsupseteq t_1 \Rightarrow F(t_2) \subseteq^* F(t_1)$;

2) $t_1 \perp t_2 \Rightarrow F(t_1) \cap F(t_2)$ is finite.

Such functions are easy to construct by induction: at successor steps, take an element of $[\omega]^\omega$ and form a size ω_1 family of almost disjoint infinite subsets of it. At limit steps, take pseudointersections. Since in V we have $\omega_1 < \mathfrak{t}$, there is some $S \in [\omega]^\omega$ satisfying

$$(\forall \alpha < \omega_1)\, S \subseteq^* F(x \restriction \alpha).$$

Since $\mathcal{P}(\omega) \subseteq M$, in particular $S \in M$. Now x can be defined in M by

$$x = \bigcup \{t \in {}^{<\omega_1}\omega_1 : S \subseteq^* F(t)\}.$$

Thus, $x \in M$, and we are done. □

We could have proved this proposition using vertical instead of horizontal coding to get the function f to have domain ${}^{\omega_1}2$. At this point, it appears that horizontal coding is strictly better than vertical coding. This next proposition shows that the methods are in fact incomparable, because the tree ${}^{<\kappa}2$ is not wide enough for horizontal coding to work. Recall that an infinite cardinal is *weakly compact* iff it is strongly inaccessible and has the tree property. The function Exit([[a]]) has the expected definition.

Proposition V.30. *Fix $a \in {}^{\kappa}2$. Fix M a transitive model of ZFC such that $\kappa \in M$, ${}^{<\kappa}2 \subseteq M$, ($\kappa$ is weakly compact)^M, and some fixed $g : ({}^{\kappa}2)^M \to \kappa$ in M satisfies*

$$(\forall x \in ({}^{\kappa}2)^M)\, \mathrm{Exit}([[a]])(x) \leq g(x),$$

then $a \in M$.

Proof. As usual define $B \subseteq {}^{<\kappa}2$ by

$$B = \{t \in {}^{<\kappa}2 : g(x) \geq \mathrm{Dom}(t) \text{ for all } x \sqsupseteq t \text{ in } M\}.$$

Let $T \subseteq B$ be the set of elements of B all of whose initial segments are also in B. Assume towards a contradiction that $a \notin M$. As usual, we can argue that T has a length κ branch (in V). Once we show T has a length κ branch in M, we will be done.

Since (κ is strongly inaccessible)^M, we have (each level of T has size $< \kappa)^M$. Since T has height κ in V, (T has height $\kappa)^M$. Combining these last two facts with the fact that (κ has the tree property)^M, we get that T has a length κ branch in M. □

We insisted that M be a model of ZFC so that we could simply state the hypothesis on κ in M. Since ω is weakly compact, this argument gives us an alternate way to handle that absoluteness portion of the proof of Theorem V.8! Note that removing the hypothesis ${}^{<\kappa}2 \subseteq M$ in the proposition above would be a disaster: we are building a path in V and we need to be sure that each proper initial segment of this path is within M (because only then is hypothesis that $a \notin M$ useful)! Finally, it would be immoral to not mention the brute force way to get the absoluteness of the existence of length κ paths through subtrees of ${}^{<\kappa}\lambda$: elementary substructures. This is different from our previous propositions because the model in question need not be transitive (and so it does not have an application to distributivity laws for complete Boolean algebras).

Proposition V.31. *For each $A \subseteq \lambda$, whenever $\langle M, \in\rangle \prec V$ with $\{\kappa, \lambda\} \cup {}^{<\kappa}\lambda \subseteq M$ and some $g : {}^{\kappa}\lambda \to \kappa$ in M satisfies*

$$(\forall x \in {}^{\kappa}\lambda)\, f_A(x) \leq g(x),$$

then $A \in M$.

The reader may easily fill in the details. Notice the hypothesis that g everywhere dominates f_A, instead of merely satisfying $(\forall x \in ({}^{\kappa}\lambda)^M)\, f_A(x) \leq g(x)$. The punch line of the proof is that elementarity allows us to conclude that from the existence of the length κ path we build in V, there must be a similar length κ path in M.

5.7 Definitions from Prewellorderings

In Section 5.6, we stated the coding results in terms of functions from ${}^{\kappa}\lambda$ to κ. When instead looking at functions from λ^{κ} to κ, we get analogous coding results at the expense of throwing in an appropriate surjection. We will give a couple examples in the case of encoding subsets of ω and encoding subsets of ${}^{\omega}\omega$.

Proposition V.32. *Let λ be a cardinal and $h : \lambda \to {}^{\omega}\omega$ be a surjection. Then there is a function $F : \mathcal{P}(\omega) \to {}^{\lambda}\omega$ definable from h such that for each $A \subseteq \omega$ and $g : \lambda \to \omega$,*

$$F(A) \leq g \Rightarrow A \text{ is definable from } g \text{ and } h.$$

Proof. For each $A \subseteq \omega$, let $f_A : {}^{\omega}\omega \to \omega$ be the horizontal encoding function from Section 5.6. Let $F : \mathcal{P}(\omega) \to {}^{\lambda}\omega$ be the function

$$F(A)(\alpha) := f_A(h(\alpha)).$$

Now fix $A \subseteq \omega$ and $g : \lambda \to \omega$ satisfying $F(A) \leq g$. As usual, way may argue that there is some $t \in {}^{<\omega}\omega$ satisfying

$$A = \{z \in \omega : (\forall \alpha < \lambda)\, h(\alpha) \upharpoonright (|t|+1) = t^\frown z \Rightarrow g(x) \geq |t|+1\},$$

and we are done. $\square$

Since the constructible universe L satisfies CH and has a definable well-ordering of ${}^{\omega}\omega$, we have the following:

Corollary V.33. *($V = L$) There is a definable function $F : \mathcal{P}(\omega) \to {}^{\omega_1}\omega$ such that for each $A \subseteq \omega$ and $g : \omega_1 \to \omega$,*

$$F(A) \leq g \Rightarrow A \text{ is definable from } g.$$

For the next higher type we have the following, whose proof we omit:

Proposition V.34. *Let λ be a cardinal and $h : \lambda \to {}^{\omega}\omega$ be a surjection. Then there is a function $F : \mathcal{P}({}^{\omega}\omega) \to {}^{\lambda}\omega$ definable from h such that for each $A \subseteq {}^{\omega}\omega$ and $g : \lambda \to \omega$,*

$$F(A) \leq g \Rightarrow A \text{ is definable from } g, h, \text{ and a real.}$$

5.8 Complete Boolean Algebras

We will now apply the coding results of Section 5.6 to obtain implications between distributivity laws for complete Boolean algebras. Throughout this section, let $\mathbb{B}$ be a complete Boolean algebra. We have the following:

Theorem V.35 (A)**.** *Let λ be an infinite cardinal. If*

1) $\mathbb{B}$ is weakly (λ^ω, ω)-distributive,

then $\mathbb{B}$ is $(\lambda, 2)$-distributive.

Theorem V.36 (B)**.** *Let κ be a weakly compact cardinal. If*

1) $\mathbb{B}$ is weakly $(2^\kappa, \kappa)$-distributive and

2) $\mathbb{B}$ is $(\alpha, 2)$-distributive for each $\alpha < \kappa$,

then $\mathbb{B}$ is $(\kappa, 2)$-distributive.

Theorem V.37 (C)**.** *If*

1) $\mathbb{B}$ *is weakly* $(2^{\omega_1}, \omega_1)$*-distributive,*

2) $\mathbb{B}$ *is* $(\omega, 2)$*-distributive, and*

3) $1 \Vdash_{\mathbb{B}} (\omega_1 < \mathbf{t})$*,*

then $\mathbb{B}$ *is* $(\omega_1, 2)$*-distributive.*

Theorem A follows from Proposition V.27, Theorem B follows from Proposition V.30, and Theorem C follows from Proposition V.29. We give the argument for Theorem A, as the other two are quite similar. The point is the following easy intermediate lemma, whose order of quantifiers is not as powerful as Proposition V.27, but the functions have the ordinal $(\lambda^\omega)^M$ instead of the set $({}^\omega\lambda)^M$ as their domains:

Lemma V.38. *Let* M *be a transitive model of ZF such that the ordinal* λ *is in* M *and* $({}^\omega\lambda)^M$ *can be well-ordered in* M*. Assume* $\mathcal{P}(\lambda) - M \neq \emptyset$*. Then there is a function* $f : (\lambda^\omega)^M \to \omega$ *which cannot be everywhere dominated by any* $g : (\lambda^\omega)^M \to \omega$ *in* M*.*

Proof. Use Proposition V.27 with any $A \in \mathcal{P}(\lambda) - M$ to get an $\tilde{f} : {}^\omega\lambda \to \omega$ such that there is no $\tilde{g} : ({}^\omega\lambda)^M \to \omega$ in M satisfying

$$(\forall x \in ({}^\omega\lambda)^M)\, \tilde{f}(x) \leq \tilde{g}(x).$$

Since $({}^\omega\lambda)^M$ can be well-ordered in M, fix a bijection

$$\eta : (\lambda^\omega)^M \to ({}^\omega\lambda)^M$$

in M. Define $f : (\lambda^\omega)^M \to \omega$ by

$$f(\alpha) := \tilde{f}(\eta(\alpha)).$$

That is, the following diagram commutes:

$$\begin{array}{ccc} ({}^{\omega}\lambda)^M & \xrightarrow{\tilde{f}} & \omega \\ \uparrow{\scriptstyle \eta} & \nearrow{\scriptstyle f} & \\ (\lambda^{\omega})^M. & & \end{array}$$

Let $g : (\lambda^{\omega})^M \to \omega$ be an arbitrary function in M. Suppose, towards a contradiction, that

$$(\forall \alpha < (\lambda^{\omega})^M)\, f(\alpha) \leq g(\alpha).$$

This implies that if we define $\tilde{g} : ({}^{\omega}\lambda)^M \to \omega$ by

$$\tilde{g}(x) := g(\eta^{-1}(x)),$$

we have that $\tilde{g} \in M$, and

$$(\forall x \in ({}^{\omega}\lambda)^M)\, \tilde{f}(x) \leq \tilde{g}(x).$$

This is a contradiction. □

We now get Theorem A. Let us show the contrapositive. Let $\mu = \lambda^{\omega}$. Suppose $\mathbb{B}$ is not $(\lambda, 2)$-distributive. Force with $\mathbb{B}$. The extension has a new subset of λ. By the lemma above (using M for the ground model and V for the extension), there is a function from μ to ω in the extension which cannot be everywhere dominated by any function in the ground model. Hence, $\mathbb{B}$ is not weakly $(\mu, 2)$-distributive.

With regard to Theorem C, we may ask if it is consistent with ZFC that every complete Boolean algebra that is both (ω, ∞)-distributive and weakly (λ, ω_1)-distributive for all λ must also be $(\omega_1, 2)$-distributive. We hope that this follows from MA or a similar axiom. Indeed, a model where this fails would appear to be pathological given the coding results we have seen. By Theorem C, we need only worry about those $\mathbb{B}$ satisfying $1 \Vdash_{\mathbb{B}} (\omega_1 = \mathfrak{t})$. The final result of this chapter will, together with Theorem C, suggest that $\mathrm{MA}(\omega_1)$ does imply this.

The main idea of this next proposition is the following: if we have a size λ collection $\mathcal{C}$ of antichains in $\mathbb{B}$ each of size κ', then if $\mathbb{B}$ is weakly (λ, κ')-distributive, then there is a maximal antichain $A \subseteq \mathbb{B}$ such that below each $a \in A$, each antichain in $\mathcal{C}$ has $< \kappa'$ non-zero elements. Assuming also that $\mathbb{B}$ is $(\omega, |\mathbb{B}|)$-distributive, we can repeatedly apply this construction countably many times until we produce a maximal antichain B_ω such that below each $b' \in B_\omega$, each antichain of $\mathbb{B}$ has only countably many non-zero elements. That is, B_ω will witness that $\mathbb{B}$ is "locally c.c.c.". Then, we will use a result of Baumgartner to conclude that since $\mathbb{B}$ is locally c.c.c. and $(\omega, 2)$-distributive, $\mathbb{B}$ is either $(\omega_1, 2)$-distributive or a Suslin tree can be embedded into $\mathbb{B}$. If we assume there are no Suslin trees (which follows from $\mathrm{MA}(\omega_1)$), we get that $\mathbb{B}$ must be $(\omega_1, 2)$-distributive. Given a complete Boolean algebra $\mathbb{B}$ and $a, b \in \mathbb{B}$, we say a is non-zero below b iff $a \wedge b \neq 0_{\mathbb{B}}$.

Proposition V.39. *Assume there are no Suslin trees. Let $\mathbb{B}$ be a complete Boolean algebra and κ be a cardinal satisfying the following:*

1) $\mathbb{B}$ is (ω, ∞)-distributive;

2) $\mathbb{B}$ is κ-c.c.;

3) $\kappa < \aleph_{\omega_1}$;

4) $\mathbb{B}$ is weakly $(|\mathbb{B}|^{\kappa'}, \kappa')$-distributive for each uncountable $\kappa' < \kappa$.

Then $\mathbb{B}$ is $(\omega_1, 2)$-distributive.

Proof. We will construct a sequence of maximal antichains

$$\langle B_n \subseteq \mathbb{B} : n \in \omega \rangle$$

such that $B_0 := \{1_{\mathbb{B}}\}$ and $(\forall n < m < \omega)\, B_m$ refines B_n. Each B_n will have the property that for any maximal antichain A below an element $b \in B_n$, for each

$b' \in B_{n+1}$ extending b, A will have $< |A|$ non-zero elements below b'. We will then define the maximal antichain B_ω to refine each B_n, and we will argue that below each $b_\omega \in B_\omega$, $\mathbb{B}$ is c.c.c.

Let $\kappa < \aleph_{\omega_1}$ be the least cardinal such that $\mathbb{B}$ is κ-c.c. Define $B_0 := \{1_{\mathbb{B}}\}$. We will now define a maximal antichain $B_1 \subseteq \mathbb{B}$ (which trivially refines B_0). Every antichain in $\mathbb{B}$ has size $< \kappa$. Consider an uncountable cardinal $\kappa' = \aleph_\alpha < \kappa$. Let $\lambda := |\mathbb{B}|^{\kappa'}$. Let $\langle A_\beta : \beta < \lambda \rangle$ be an enumeration of the maximal antichains in $\mathbb{B}$ of size κ'. For each $\beta < \lambda$, let $\langle a_{\beta,\gamma} : \gamma < \kappa' \rangle$ be an enumeration of the elements of A_β. Let $\dot{G}$ be the canonical name for the generic filter. Fix a name $\dot{f}$ such that $1 \Vdash \dot{f} : \check{\lambda} \to \check{\kappa}'$ and

$$1 \Vdash (\forall \beta < \check{\lambda})\, \check{a}_{\beta, \dot{f}(\beta)} \in \dot{G}.$$

By hypothesis, $\mathbb{B}$ is weakly (λ, κ')-distributive, so there is a maximal antichain $C_{0,\alpha} \subseteq \mathbb{B}$ (which trivially refines B_0) and for each $c \in C_{0,\alpha}$ a function $g_c : \lambda \to \kappa'$ such that $c \Vdash \dot{f} \leq \check{g}_c$. Hence, for each $c \in C_{0,\alpha}$,

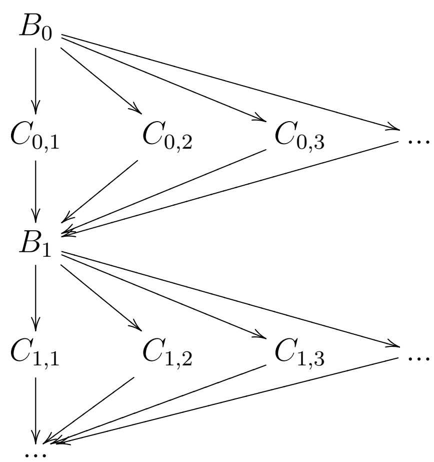
$$c \Vdash (\forall \beta < \check{\lambda})(\forall \gamma < \check{\kappa}')\, \gamma > \check{g}_c(\beta) \Rightarrow \check{a}_{\beta,\gamma} \notin \dot{G}.$$

This implies that for each A_β, below each $c \in C_{0,\alpha}$ there are $< |A_\beta|$ non-zero elements of A_β.

For each $\aleph_\alpha < \kappa$, we have such a maximal antichain $C_{0,\alpha} \subseteq \mathbb{B}$. Since $\kappa < \aleph_{\omega_1}$, the family $\langle C_{0,\alpha} \subseteq \mathbb{B} : \aleph_\alpha < \kappa \rangle$ is countable. Since $\mathbb{B}$ is (ω, ∞)-distributive, we may fix a single maximal antichain $B_1 \subseteq \mathbb{B}$ which refines every $C_{0,\alpha}$. Note that B_1 has the property that for each maximal antichain $A \subseteq \mathbb{B}$ (below $1_{\mathbb{B}}$) and $b' \in B_1$, A has $< |A|$ non-zero elements below b'.

We will now define B_2. Consider an uncountable cardinal $\kappa' = \aleph_\alpha < \kappa$. Let $\lambda := |\mathbb{B}|^{\kappa'}$. Let $\langle A_\beta : \beta < \lambda \rangle$ be an enumeration of all size κ' antichains that are each a partition of some element of B_1. Since $\mathbb{B}$ is weakly (λ, κ')-distributive, we may use an argument similar to before to produce a maximal antichain $C_{1,\alpha}$ which refines B_1 such that for each A_β has $< |A_\beta|$ non-zero elements below each $c \in C_{1,\alpha}$. This completes the construction of $C_{1,\alpha}$. Like before, we may use the (ω, ∞)-distributivity of $\mathbb{B}$ to get a common refinement B_2 of every maximal antichain in the family $\langle C_{1,\alpha} : \aleph_\alpha < \kappa \rangle$. Note that B_2 has the property that for each partition A of some element of B_1 and $b' \in B_2$, A has $< |A|$ non-zero elements below b'.

We may continue this procedure to get a sequence $\langle B_n : n \in \omega \rangle$ of maximal antichains of $\mathbb{B}$. The following diagram depicts the maximal antichains which we have constructed, where an arrow represents refinement:



Using the (ω, ∞)-distributivity of $\mathbb{B}$ once more, we may get a single maximal antichain $B_\omega \subseteq \mathbb{B}$ which refines each B_n. We will now argue that given any maximal antichain $A \subseteq \mathbb{B}$ and $b_\omega \in B_\omega$, A has only countably many non-zero elements below b.

Fix an arbitrary maximal antichain $A_0 \subseteq \mathbb{B}$. Fix $b_\omega \in B_\omega$. Let $\kappa_0 := |A_0|$. If $\kappa_0 \leq \omega$, we are done. If not, let b_1 be the unique element of B_1 above b_ω. By the construction of B_1, A_0 has $< \kappa_0$ non-zero elements below b_1. Let $\kappa_1 < \kappa_0$ be the number of such non-zero elements. That is, letting

$$A_1 := \{a \wedge b_1 : a \in A_0\},$$

we have $|A_1| = \kappa_1 < \kappa_0$. If $\kappa_1 \leq \omega$, we are done because $|\{a \wedge b_\omega : a \in A_0\}| \leq |A_1| \leq \omega$. Otherwise, let b_2 be the unique element of B_2 above b_ω. By the construction of B_2, A_1 has $< \kappa_1$ non-zero elements below b_2. Let $\kappa_2 < \kappa_1$ be the number of such non-zero elements. That is, letting

$$A_2 := \{a \wedge b_2 : a \in A_1\},$$

we have $|A_2| = \kappa_2 < \kappa_1$. If $\kappa_2 \leq \omega$, we are done by similar reasons as before. If not, then we may continue the procedure. However, the procedure will eventually terminate. This is because if not, then we would have an infinite decreasing sequence of cardinals

$$\kappa_0 > \kappa_1 > \kappa_2 > ...,$$

which is impossible. Thus, A_0 has only countably many non-zero elements below b_ω.

At this point, we have argued that below the maximal antichain B_ω, $\mathbb{B}$ has the c.c.c. Now, it must be that $\mathbb{B}$ is $(\omega_1, 2)$-distributive. Let us explain. It suffices to show that $\mathbb{B}$ is $(\omega_1, 2)$-distributive below each element of B_ω. Fix any $b_\omega \in B_\omega$. Below b_ω, $\mathbb{B}$ is c.c.c. and $(\omega, 2)$-distributive. Suppose, towards a contradiction, that $\mathbb{B}$ is not $(\omega_1, 2)$-distributive. Quoting a result of Baumgartner [1], there exists a Suslin tree which, when turned upside down, can be embedded into $\mathbb{B}$ below b_ω. However, we assumed there are no Suslin trees. This completes the proof. □

[1] This was discovered independently by Andreas Blass who was told it was already proved by James Baumgartner. However, neither the author nor Blass have been able to find a proof in the literature.

CHAPTER VI

Impossibility of Coding for Pointwise Eventual Domination

The purpose of this chapter is to discuss obstructions to computing the cofinality of $\mathcal{B}_\alpha({}^\omega\omega, \leq^*)$ for $\alpha \geq 1$. It will become clear that the methods of the previous chapter do not suffice. Within the next chapter we will successfully perform the computation by proving a strong infinite coding theorem.

In the first section, we observe that it is consistent with ZFC that $\operatorname{cf}\operatorname{All}({}^\omega\omega, \leq^*) < 2^{2^\omega}$. This tells us that a ZFC proof that $\operatorname{cf}\mathcal{B}_\alpha({}^\omega\omega, \leq^*) = 2^\omega$ for $\alpha \geq 1$ must be substantially different from our proof that $\operatorname{cf}\mathcal{B}_\alpha(\omega, \leq) = 2^\omega$, because the latter proof generalized easily (Corollary V.21) to show that $\operatorname{cf}\operatorname{All}(\omega, \leq) = 2^{2^\omega}$. We have an impossibility of coding result, in the sense that ZFC *cannot* prove the following: for each $A \subseteq {}^\omega\omega$, Alice can produce a function $f : {}^\omega\omega \to {}^\omega\omega$ such that if $g : {}^\omega\omega \to {}^\omega\omega$ pointwise eventually dominates f, then Bob can guess A from g using only continuum many guesses.

In the second section, we show that the simplest (in some sense) encoding scheme (which we call "Naive Vertical Coding") to try to show $\operatorname{cf}\mathcal{B}_\alpha({}^\omega\omega, \leq^*) = 2^\omega$ (for $\alpha \geq 1$) is doomed to fail. Specifically, if for each $A \subseteq \omega$ we assign a function $f : {}^\omega\omega \to {}^\omega\omega$ with the property that

$$(\forall k \in \omega)(\exists a \in {}^\omega\omega)\, f(x)(k) = \operatorname{Exit}([[a]])(x),$$

then A need not be constructible from any code for a Borel function g satisfying $f \le^* g$. Hence, A need not be Δ^1_2 in a Borel code for such a g. This is convincing evidence that such an encoding scheme cannot work, because a countable set of guesses for A from (a code c for) the Borel function g is likely to be a subset of $\mathcal{P}(\omega) \cap L[c]$. The reason for us considering "constructible from" is because we will use forcing to get our counterexample: the generic real A will not be constructible from any real in the ground model, and yet the function f associated to A will be pointwise eventually dominated by a Borel function with a code in the ground model.

In the third section, we will show that an infinite coding theorem to prove $\operatorname{cf} \mathcal{B}_\alpha({}^\omega\omega, \le^*) = 2^\omega$ (for $\alpha \ge 1$) must be specific to Borel functions, and cannot (in ZFC) generalize to projective functions. This is because of the consistent existence of a projective well-ordering of ${}^\omega\omega$ together with $\omega_2 \le \mathfrak{b}$. In the final section, we show what can go wrong when considering relations significantly weaker than pointwise eventual domination.

6.1 Considering All Functions

The point of this section is to investigate the poset $\mathrm{All}({}^\omega\omega, \le^*)$ of *all* functions from ${}^\omega\omega$ to ${}^\omega\omega$ ordered by pointwise eventual domination. We will show that it is qualitatively different than the poset $\mathrm{All}(\omega, \le)$ of all functions from ${}^\omega\omega$ to ω ordered pointwise. The slogan is as follows: arbitrary subsets of ${}^\omega\omega$ *can* be encoded into elements of $\mathrm{All}(\omega, \le)$, but *cannot* (in ZFC) be encoded into elements of $\mathrm{All}({}^\omega\omega, \le^*)$.

For the rest of this section, we will use the symbol $\mathfrak{c}$ to denote 2^ω. Let $\le_{def({}^\omega\omega)}$ be the binary relation defined by $A \le_{def({}^\omega\omega)} B$ iff A is definable in the language of set theory by a formula using only B and real numbers as parameters. [1] Note that

[1]Technically, $\le_{def({}^\omega\omega)}$ is not definable by Tarski's undefinability of truth, but by restricting quantifiers to a

for each $B \subseteq {}^\omega\omega$, the set $\{A \subseteq {}^\omega\omega : A \leq_{def({}^\omega\omega)} B\}$ has size 2^ω. By the results in the previous chapter, there is a morphism from $\mathrm{All}(\omega, \leq)$ to $\langle \mathcal{P}({}^\omega\omega), \leq_{def({}^\omega\omega)}\rangle$. This implies

$$\mathrm{cf}\,\mathrm{All}(\omega, \leq) = 2^{\mathfrak{c}}.$$

On the other hand, we will soon show that there can be no ZFC proof that there is a morphism from $\mathrm{All}({}^\omega\omega, \leq^*)$ to $\langle \mathcal{P}({}^\omega\omega), \leq_{def({}^\omega\omega)}\rangle$. We will prove this by constructing a model of ZFC in which

$$\mathrm{cf}\,\mathrm{All}({}^\omega\omega, \leq^*) < 2^{\mathfrak{c}}.$$

The idea is to build a model in which simultaneously there is a scale in $\langle {}^\omega\omega, \leq^*\rangle$ of length $\mathfrak{c}$ and $\mathrm{cf}\,\langle {}^{\mathfrak{c}}\mathfrak{c}, \leq^*\rangle < 2^{\mathfrak{c}}$.

Observation VI.1. *Let $\langle \phi_-, \phi_+\rangle$ be a morphism from a poset $\mathbb{P}$ to a poset $\mathbb{Q}$. Let λ be an infinite cardinal. Let $\mathbb{P}'$ be the poset of functions from λ to $\mathbb{P}$ ordered pointwise. Let $\mathbb{Q}'$ be defined similarly. Then there is a morphism $\langle \phi'_-, \phi'_+\rangle$ from $\mathbb{P}'$ to $\mathbb{Q}'$.*

Proof. Define $\phi'_- : {}^\lambda\mathbb{Q} \to {}^\lambda\mathbb{P}$ and $\phi'_+ : {}^\lambda\mathbb{P} \to {}^\lambda\mathbb{Q}$ as follows:

$$\phi'_-(g) := x \mapsto \phi_-(g(x))$$

$$\phi'_+(f) := x \mapsto \phi_+(f(x)).$$

The pair $\langle \phi'_-, \phi'_+\rangle$ is as desired. □

Combining this with Observations I.9 and I.10, we get the following corollaries.

Corollary VI.2. *If there is an unbounded chain in $\langle {}^\omega\omega, \leq^*\rangle$ of length a regular cardinal κ, then in addition to $\kappa \leq \mathfrak{d}$ we have*

$$\mathrm{cf}\,\langle {}^{\mathfrak{c}}\kappa, \leq\rangle \leq \mathrm{cf}\,\mathrm{All}({}^\omega\omega, \leq^*).$$

---

sufficiently large initial segment of V we can avoid this problem.

Corollary VI.3. *If there is a scale in $\langle {}^\omega\omega, \le^* \rangle$ of length κ (which must be a regular cardinal), then in addition to $\kappa = \mathfrak{b} = \mathfrak{d}$ we have*

$$\mathrm{cf}\,\langle {}^{\mathfrak{c}}\kappa, \le \rangle = \mathrm{cf}\,\mathrm{All}({}^\omega\omega, \le^*).$$

Of course, there is an unbounded chain in $\langle {}^\omega\omega, \le^* \rangle$ of length $\mathfrak{b}$, so we have

$$\mathrm{cf}\,\langle {}^{\mathfrak{c}}\mathfrak{b}, \le \rangle \le \mathrm{cf}\,\mathrm{All}({}^\omega\omega, \le^*). \tag{6.1}$$

Let κ be a regular cardinal. Proposition II.1 shows that $\mathrm{cf}\,\langle {}^{\mathfrak{c}}\kappa, \le \rangle \ge \mathfrak{c}^+$. Hence, $2^{\mathfrak{c}} = \mathfrak{c}^+$ (and therefore GCH) implies $\mathrm{cf}\,\langle {}^{\mathfrak{c}}\kappa, \le \rangle = 2^{\mathfrak{c}}$. The following is a more interesting implication:

Corollary VI.4. *If $2^{\mathfrak{b}} = \mathfrak{c}$, then $\mathrm{cf}\,\mathrm{All}({}^\omega\omega, \le^*) = 2^{\mathfrak{c}}$.*

Proof. Let $\lambda = \mathfrak{c}$ and $\kappa = \mathfrak{b}$. We have $\lambda^\kappa = (2^\omega)^{\mathfrak{b}} = 2^{\mathfrak{b}} = \mathfrak{c} = \lambda$, so by Corollary II.27, $\mathrm{cf}\,\langle {}^{\mathfrak{c}}\mathfrak{b}, \le \rangle = 2^{\mathfrak{c}}$. The result follows by the inequality (6.1). $\square$

Of course, $2^{\mathfrak{b}} = \mathfrak{c}$ implies $\mathfrak{b} < \mathfrak{c}$. There are three cases:

1) $2^{\mathfrak{b}} = \mathfrak{c}$;

2) $\mathfrak{b} = \mathfrak{c}$;

3) $\mathfrak{b} < \mathfrak{c} < 2^{\mathfrak{b}}$.

The corollary above handles the first case. The second case implies $\mathfrak{b} = \mathfrak{d} = \mathfrak{c}$, which in turn implies there is a scale in $\langle {}^\omega\omega, \le^* \rangle$ of length $\mathfrak{c}$. This, by Corollary VI.3, reduces the problem to studying the poset $\langle {}^{\mathfrak{c}}\mathfrak{c}, \le \rangle$ (and in this case $\mathfrak{c}$ is regular). In particular,

$$\mathfrak{b} = \mathfrak{c} \text{ and } \mathrm{cf}\,\langle {}^{\mathfrak{c}}\mathfrak{c}, \le \rangle < 2^{\mathfrak{c}} \Rightarrow \mathrm{cf}\,\mathrm{All}({}^\omega\omega, \le^*) < 2^{\mathfrak{c}}. \tag{6.2}$$

We will now build a model of ZFC satisfying the left-hand side of (6.2).

Recall Theorem II.10 (due to Cummings and Shelah), which gives us that if λ is a regular cardinal satisfying $\lambda^{<\lambda} = \lambda$ and $\mathbb{Q}$ is a poset in which every size λ subset is bounded, then there is a λ-closed (meaning closed under sequences of length $< \lambda$) and λ^+-c.c. forcing $\mathbb{D}(\lambda, \mathbb{Q})$ such that $1 \Vdash (\operatorname{cf} \langle {}^{\check{\lambda}}\check{\lambda}, \leq \rangle = \check{\delta})$ where $\delta = \operatorname{cf} \mathbb{Q}$.

Suppose we start with a ground model satisfying $\mathfrak{b} = \mathfrak{c}$, $\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$, and $\mathfrak{c}^+ < 2^{\mathfrak{c}}$. Let $\lambda := \mathfrak{c}$ and $\mathbb{Q} := \langle \lambda^+, \leq \rangle$. When we force with $\mathbb{D}(\lambda, \mathbb{Q})$, in the extension we will have $\operatorname{cf} \langle {}^{\mathfrak{c}}\mathfrak{c}, \leq \rangle = \lambda^+ < 2^{\mathfrak{c}}$. We will also have $\mathfrak{b} = \mathfrak{c}$, but this relies on the fact that the forcing is λ-closed. Indeed, simply not adding reals and not collapsing cardinals does not suffice to preserve $\mathfrak{b} = \mathfrak{c}$, as is shown in [36]. To get simultaneously $\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$ and $\mathfrak{c}^+ < 2^{\mathfrak{c}}$, the tower number $\mathfrak{t}$ is useful.

Recall that $\mathfrak{t}$ is the minimum length of an unbounded chain in $\langle [\omega]^\omega, \supseteq^* \rangle$. A useful fact about $\mathfrak{t}$ is $2^{<\mathfrak{t}} = \mathfrak{c}$ (see [2] for a proof). This implies $\mathfrak{c}^{<\mathfrak{t}} = \mathfrak{c}$. Also, $\mathfrak{t}$ is regular and $\mathfrak{t} \leq \mathfrak{b}$. We will need the following simple observation (which can be made much more general but there is no need here):

Observation VI.5. *If $\mathbb{P}$ is a forcing that is $\mathfrak{c}$-closed and $\mathfrak{t} = \mathfrak{c}$, then $1 \Vdash (\mathfrak{t} = \mathfrak{c})$.*

Proof. Let $\lambda = \mathfrak{c}$. Since $\mathbb{P}$ is $\mathfrak{c}$-closed, it does not add reals, so $1 \Vdash ([\omega]^\omega = \check{\overbrace{[\omega]^\omega}})$. Additionally since $\mathbb{P}$ is $\mathfrak{c}$-closed, cardinals $\leq \lambda$ are preserved, so $1 \Vdash (\mathfrak{c} = \check{\lambda})$. Suppose, towards a contradiction, that $1 \nVdash (\mathfrak{t} = \mathfrak{c})$. There must be $p \in \mathbb{P}$ and a name $\dot{\tau}$ satisfying $p \Vdash (\dot{\tau}$ is an unbounded chain in $\langle [\omega]^\omega, \supseteq^* \rangle$ of length $< \check{\lambda})$. This is a contradiction, because $\mathbb{P}$ does not add sequences of length $< \mathfrak{c}$ whose elements are in the ground model. □

We now have all the pieces for the promised consistency result. Recall from [33] that $\operatorname{Fn}(I, J, \lambda)$ is the poset of partial functions from I to J of size $< \lambda$ ordered by extension. By Lemma 6.10 of [33], the forcing $\operatorname{Fn}(I, J, \lambda)$ has the $(|J|^{<\lambda})^+$-c.c. When

$J = 2$ and $|I| \geq \lambda$, $\mathrm{Fn}(I, J, \lambda)$ is the forcing to add $|I|$ *Cohen* subsets of λ. In this case, it is also called $\mathrm{Add}(\lambda, |I|)$.

Proposition VI.6. *There is a forcing extension in which* $\mathfrak{b} = \mathfrak{c}$ *and* $\mathrm{cf}\,\langle {}^{\mathfrak{c}}\mathfrak{c}, \leq \rangle < 2^{\mathfrak{c}}$*, so therefore*

$$\mathrm{cf}\,\mathrm{All}({}^{\omega}\omega, \leq^*) < 2^{\mathfrak{c}}.$$

Proof. By (6.1), it suffices to force both $\mathfrak{b} = \mathfrak{c}$ and $\mathrm{cf}\,\langle {}^{\mathfrak{c}}\mathfrak{c}, \leq \rangle < 2^{\mathfrak{c}}$. Without loss of generality, assume $\mathfrak{t} = \mathfrak{c}$ holds in $M_1 := V$ (we can always force Martin's Axiom, which implies this). Since $\mathfrak{t}$ is regular, so is $\mathfrak{c}$. We will first construct a forcing extension M_2 of M_1 which satisfies the following:

1) $\mathfrak{t} = \mathfrak{c}$;

2) $\mathfrak{c}$ is regular;

3) $\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$;

4) $\mathfrak{c}^{+} < 2^{\mathfrak{c}}$.

Notice that 1) implies 2) and 3). Let M_2 be a forcing extension of M_1 obtained by adding $\mathfrak{c}^{++}$ *Cohen* subsets of $\mathfrak{c}$. That is, the forcing $\mathbb{P}$ which consists of partial functions from $\mathfrak{c} \times \mathfrak{c}^{++}$ to $\{0, 1\}$ of size $< \mathfrak{c}$ (ordered by end-extension):

$$\mathbb{P} = \mathrm{Fn}(\mathfrak{c} \times \mathfrak{c}^{++}, 2, \mathfrak{c}).$$

Since this forcing is $\mathfrak{c}$-closed and $\mathfrak{t} = \mathfrak{c}$, by Observation VI.5 we have that M_2 satisfies 1). Also, by the nature of this forcing, M_2 satisfies 4). Hence, M_2 satisfies 1) through 4). Since $2^{<\mathfrak{c}} = \mathfrak{c}$ (because $\mathfrak{t} = \mathfrak{c}$), $\mathbb{P}$ has the $\mathfrak{c}^{+}$-c.c., so cardinals $> \mathfrak{c}$ as preserved. Since $\mathbb{P}$ is $\mathfrak{c}$-closed, cardinals $\leq \mathfrak{c}$ are preserved as well.

Let $\lambda := \mathfrak{c}^{M_2} = \mathfrak{c}$ and $\delta := (\lambda^{+})^{M_2} = \lambda^{+}$. By 1) through 4), we have $(\mathfrak{t} = \lambda)^{M_2}$, $(\lambda$ is regular$)^{M_2}$, $(\lambda^{<\lambda} = 2^{\lambda})^{M_2}$, and $(\lambda^{+} < 2^{\lambda})^{M_2}$. Within M_2 define $\mathbb{Q} := \langle \lambda^{+}, \leq \rangle$. Of course,

$$(\mathfrak{b}(\mathbb{Q}) = \operatorname{cf}\mathbb{Q} = \delta < 2^{\lambda})^{M_2}.$$

Within M_2 consider $\mathbb{D}(\lambda, \mathbb{Q})$. Let M_3 be a forcing extension of M_2 by $\mathbb{D}(\lambda, \mathbb{Q})$. Since $(\mathbb{D}(\lambda, \mathbb{Q})$ is λ-closed$)^{M_2}$, $\mathfrak{c}^{M_3} = \lambda$. By 4) of Theorem II.10,

$$(\operatorname{cf}\langle {}^{\lambda}\lambda, \leq\rangle = \delta)^{M_3}.$$

Since $(\mathbb{D}(\lambda, \mathbb{Q})$ is λ-closed and λ^+-c.c.$)^{M_2}$, we have $(2^{\lambda})^{M_2} = (2^{\lambda})^{M_3}$, which implies

$$(\delta < 2^{\lambda})^{M_3}.$$

Thus,

$$(\operatorname{cf}\langle {}^{\mathfrak{c}}\mathfrak{c}, \leq\rangle < 2^{\mathfrak{c}})^{M_3}.$$

Since $(\mathbb{D}(\lambda, \mathbb{Q})$ is $\mathfrak{c}$-closed$)^{M_2}$ and $(\mathfrak{t} = \mathfrak{c})^{M_2}$, by Observation VI.5 we have $(\mathfrak{t} = \mathfrak{c})^{M_3}$, and so

$$(\mathfrak{b} = \mathfrak{c})^{M_3}.$$

This completes the proof. $\square$

What remains at this point is to investigate the situation when $\mathfrak{b} < \mathfrak{c} < 2^{\mathfrak{b}}$. We will content ourselves by showing $\operatorname{cf}\operatorname{All}({}^{\omega}\omega, \leq^*) = 2^{\mathfrak{c}}$ in the natural model one would construct in which $\mathfrak{b} < \mathfrak{c} < 2^{\mathfrak{b}}$. The reader may skip the rest of this section with no loss of continuity. The following lemma (which can be made much more general) deals with the main technicality. The argument is essentially the same as the one which shows that $\operatorname{Fn}(\kappa, \omega, \omega)$ forces $\mathfrak{d} = \kappa$.

Lemma VI.7. *Let* $\mathbb{P} := \operatorname{Fn}(\omega_1 \times \omega_3, \omega_1, \omega_1)$. *Assume* $\mathbb{P}$ *has the* ω_3*-c.c. Let* $\dot{G}$ *be the canonical name for the generic, so* $1 \Vdash (\dot{G} : \omega_1 \times \omega_3 \to \omega_1)$. *Let* $p \in \mathbb{P}$ *and* $\dot{\tau} \in V^{\mathbb{P}}$

satisfy $p \Vdash (\dot{\tau} : \omega_1 \times \omega_2 \to \omega_1)$. *Then there is some* $\gamma < \omega_3$ *such that* $p \Vdash$ (*no column of* $\dot{\tau}$ *can everywhere dominate the* $\check{\gamma}$*-th column of* $\dot{G}$). *That is,*

$$p \Vdash (\forall \beta < \omega_2)(\exists \alpha < \omega_1)\, \dot{G}(\alpha, \check{\gamma}) > \dot{\tau}(\alpha, \beta).$$

Proof. First, note that $\mathbb{P}$ does not collapse any cardinals. Without loss of generality, $\dot{\tau}$ is a nice name. That is,

$$\dot{\tau} := \bigcup \{ \{ \overset{\vee}{\overbrace{((\alpha, \beta), v)}} \} \times A_{\alpha,\beta,v} : \alpha < \omega_1, \beta < \omega_2, v < \omega_1 \},$$

where each $A_{\alpha,\beta,v}$ is an antichain in $\mathbb{P}$. Since $\mathbb{P}$ has the ω_3-c.c., each $A_{\alpha,\beta,v}$ has size $\leq \omega_2$. Thus, we may fix some $\gamma < \omega_3$ satisfying

$$(\forall \alpha < \omega_1)(\forall \beta < \omega_2)(\forall v < \omega_1)(\forall f \in A_{\alpha,\beta,v})\, \mathrm{Dom}(f) \subseteq \omega_1 \times \gamma.$$

That is, all of the domains of the functions in all antichains involved with the nice name $\dot{\tau}$ are to the left of the γ-th column of $\omega_1 \times \omega_3$. Informally, this implies that when we pass to a condition stronger than p to control the behavior of $\dot{\tau}$ in the extension, we can do so without imposing any additional requirements on the γ-th column of $\dot{G}$.

We claim that $p \Vdash$ (no column of $\dot{\tau}$ can everywhere dominate the $\check{\gamma}$-th column of $\dot{G}$). Suppose, towards a contradiction, that this is false. Let $p_1 \leq p$ and $\beta < \omega_2$ satisfy $p_1 \Vdash$ (the $\check{\beta}$-th column of $\dot{\tau}$ everywhere dominates the $\check{\gamma}$-th column of $\dot{G}$). That is,

$$p_1 \Vdash (\forall \alpha < \omega_1)\, \dot{G}(\alpha, \check{\gamma}) \leq \dot{\tau}(\alpha, \check{\beta}).$$

Fix $\alpha < \omega_1$ such that $(\alpha, \gamma) \notin \mathrm{Dom}(p_1)$. Now, strengthen p_1 to a condition p_2 so that p_2 decides $\dot{\tau}(\check{\alpha}, \check{\beta})$ to be some fixed value $v < \omega_1$ and

$$\mathrm{Dom}(p_1) \cap (\omega_1 \times (\omega_3 - \gamma)) = \mathrm{Dom}(p_2) \cap (\omega_1 \times (\omega_3 - \gamma).$$

That is, every element of $\text{Dom}(p_2) - \text{Dom}(p_1)$ is strictly to the left of the γ-th column of $\omega_1 \times \omega_3$.

Finally, let

$$p_3 := p_2 \cup \{((\alpha, \beta), v + 1)\}.$$

Hence, $p_3 \leq p_2$ and $p_3 \Vdash \dot{G}(\check{\alpha}, \check{\beta}) = \check{v} + 1$. We now have a contradiction, because

$$p_3 \Vdash \check{v} + 1 = \dot{G}(\check{\alpha}, \check{\gamma}) \leq \dot{\tau}(\check{\alpha}, \check{\beta}) = \check{v},$$

which is impossible. □

We can now prove the following. The proof is routine, but we include all the details to be careful.

Proposition VI.8. *There is a forcing extension in which*

$$\mathfrak{b} < \mathfrak{c} < 2^{\mathfrak{b}}$$

and

$$\text{cf}\,\text{All}({}^{\omega}\omega, \leq^*) = 2^{\mathfrak{c}}.$$

Proof. Let $\mathbb{P} := \text{Fn}(\omega_1 \times \omega_3, \omega_1, \omega_1)$. Without loss of generality, assume GCH (we can get this by forcing). Because of GCH, we have $|\mathbb{P}| = \omega_3$, $\mathbb{P}$ has the ω_3-c.c. and $\omega_3^{\omega_2} = \omega_3$. Let $M_1 := V$. Note that $\mathbb{P}$ does not add reals or collapse cardinals. Let M_2 be a forcing extension of M_1 by $\mathbb{P}$. By the nature of $\mathbb{P}$,

$$(2^{\omega_1} = \omega_3)^{M_2}.$$

Also,

$$(2^{\omega_2} = \omega_3)^{M_2}$$

(because there are $(\omega_3^{\omega_2})^{\omega_2} = \omega_3$ $\mathbb{P}$-nice names for subsets of ω_2). Let $\mathbb{Q} \in M_2$ be such that ($\mathbb{Q}$ is the forcing to add ω_2 Cohen reals)^{M_2}. Let M_3 be a forcing extension of M_2

by $\mathbb{Q}$. M_3 will be our desired model. Note that ($\mathbb{Q}$ does not collapse cardinals)^{M_2}. Also, ($|\mathbb{Q}| = \omega_2$ and $\mathbb{Q}$ has the ω_1-c.c.)^{M_2}, which implies (the number of $\mathbb{Q}$-nice names for subsets of ω_1 is at most $|^{\omega_1}({}^{\omega}\omega_2)| = {\omega_2}^{\omega_1} \leq \omega_2^{\omega_2} = 2^{\omega_2} = \omega_3)^{M_3}$, so

$$(2^{\omega_1} = \omega_3)^{M_3}.$$

By a similar argument,

$$(2^{\omega_2} = \omega_3)^{M_3}.$$

Since $\mathbb{P}$ does not add any reals, $(\mathfrak{b} = \omega_1)^{M_2}$. Since ($\mathbb{Q}$ is the forcing to add ω_2 Cohen reals)^{M_2}, also

$$(\mathfrak{b} = \omega_1)^{M_3}$$

and

$$(\mathfrak{c} = \omega_2)^{M_3}.$$

Thus, we have

$$(\omega_1 = \mathfrak{b} < \mathfrak{c} < 2^{\mathfrak{b}})^{M_3}.$$

By the above lemma applied in M_1 to $\mathbb{P}$, we have $(\mathrm{cf}\,\langle {}^{\omega_1}\omega_1, \leq\rangle = \omega_3)^{M_2}$. Hence, $(\mathrm{cf}\,\langle {}^{\omega_2}\omega_1, \leq\rangle = \omega_3)^{M_2}$. Applying Corollary II.38 in M_2 using $\lambda = \omega_2$ and $\kappa = \omega_1$, we have

$$(\mathrm{cf}\,\langle {}^{\omega_2}\omega_1, \leq\rangle = \omega_3)^{M_3}.$$

Since (there is an unbounded chain in $\langle {}^{\omega}\omega, \leq^*\rangle$ of length $\mathfrak{b} = \omega_1)^{M_3}$ and $(\omega_2 = \mathfrak{c})^{M_3}$, we may apply Corollary VI.2 to get

$$(\mathrm{cf}\,\langle {}^{\omega_2}\omega_1, \leq\rangle \leq \mathrm{cf}\,\mathrm{All}({}^{\omega}\omega, \leq^*))^{M_3}.$$

Thus, we have shown

$$(\omega_3 = \mathrm{cf}\,\langle {}^{\omega_2}\omega_1, \leq\rangle \leq \mathrm{cf}\,\mathrm{All}({}^{\omega}\omega, \leq^*) \leq 2^{\mathfrak{c}} = 2^{\omega_2} = \omega_3)^{M_3},$$

and so

$$(\operatorname{cf}\operatorname{All}({}^{\omega}\omega, \leq^*) = 2^{\mathfrak{c}})^{M_3},$$

so we are done. $\square$

If we want to modify the above argument to get a model in which $\mathfrak{b} < \mathfrak{c} < 2^{\mathfrak{b}}$ and yet $\operatorname{cf}\operatorname{All}({}^{\omega}\omega, \leq^*) < 2^{\mathfrak{c}}$, we would need to *gently* add subsets of ω_1. Adding Cohen subsets of ω_1 is somewhat violent. There seems to be no analogue of *random* reals for subsets of ω_1, and adding *Sacks* subsets of ω_1 is not as gentle as one might expect.

The proofs in this section yield much more general results, which we will state now without proof. In all these propositions, let $\mathbb{P} = \langle P, \leq_P \rangle$ be a poset, λ be an infinite cardinal, $\kappa \leq \lambda$ be a regular cardinal, and $\mathbb{P}' = \langle {}^{\lambda}P, \leq_{{}^{\lambda}P} \rangle$ be the poset of all functions from λ to P ordered pointwise by $\leq_P$. In this section, we investigated the situation where $\langle P, \leq_P \rangle = \langle {}^{\omega}\omega, \leq^* \rangle$ and $\lambda = \mathfrak{c}$.

Proposition VI.9. *If there is an unbounded chain in $\langle P, \leq_P \rangle$ of length κ, then in addition to $\kappa \leq \operatorname{cf}\langle P, \leq_P \rangle$ we have*

$$\operatorname{cf}\langle {}^{\lambda}\kappa, \leq \rangle \leq \operatorname{cf}\mathbb{P}'.$$

Proposition VI.10. *If there is a scale in $\langle P, \leq_P \rangle$ of length κ, then in addition to $\kappa = \mathfrak{b}\langle P, \leq_P \rangle = \operatorname{cf}\langle P, \leq_P \rangle$ we have*

$$\operatorname{cf}\langle {}^{\lambda}\kappa, \leq \rangle = \operatorname{cf}\mathbb{P}'.$$

Proposition VI.11. *Let $\kappa = \mathfrak{b}\langle P, \leq_P \rangle$ (so κ is regular). Assume $|P| \leq 2^{\lambda}$ (so that $|{}^{\lambda}P| = 2^{\lambda}$). If $\lambda^{\kappa} = \lambda$, then $\operatorname{cf}\mathbb{P}' = 2^{\lambda}$.*

Assume now that $\lambda = \mathfrak{c}$ and that both P and $\leq_P$ are Borel (so we may talk about $\langle P, \leq_P \rangle^M$ in any transitive model M of ZFC).

Proposition VI.12. *If it is provable in ZFC that* $\mathfrak{t} \leq \mathfrak{b}\langle P, \leq_P \rangle$ *and* $\mathrm{cf}\langle P, \leq_P \rangle \leq \mathfrak{c}$, *then there is a forcing extension in which* $\mathfrak{t} = \mathfrak{c}$ *and* $\mathrm{cf}\,\mathbb{P}' < 2^{\mathfrak{c}}$.

Proposition VI.8 is a bit too delicate to generalize in an easy to state way. Here is the natural way to generalize the proof: first, start with a model in which $\mathfrak{b}\langle P, \leq_P \rangle$ is equal to the cardinal κ. Next, add Cohen subsets of κ to make 2^κ at least κ^{++}. Finally, add real numbers by a κ-c.c. forcing in a way to keep $\kappa = \mathfrak{b}\langle P, \leq_P \rangle$ in the extension while making $\mathfrak{c}$ strictly between κ and 2^κ.

6.2 Impossibility of Naive Vertical Coding

In this section we will use Sacks forcing, so the reader may want to quickly read Section C for terminology and the basic lemmas. Let us quickly review some definitions. Given a tree $T \subseteq {}^{<\omega}\omega$, $\mathrm{Exit}(T) : {}^{\omega}\omega \to \omega$ is the function

$$\mathrm{Exit}(T)(x) := \begin{cases} 0 & \text{if } x \in [T], \\ \min\{l : x \restriction l \notin T\} & \text{otherwise.} \end{cases}$$

Given $x' \in {}^{\omega}\omega$, $[[x']] \subseteq {}^{<\omega}\omega$ is the set

$$[[x']] := \{x' \restriction l : l \in \omega\}.$$

Hence,

$$\mathrm{Exit}([[x']])(x) = \begin{cases} 0 & \text{if } x = x', \\ \min\{l : x(l-1) \neq x'(l-1)\} & \text{otherwise.} \end{cases}$$

That is, $\mathrm{Exit}([[x']])(x)$ is the level at which x deviates from x'.

The prototypical result of the last chapter is that if M is a transitive model of ZFC and $x' \in {}^{\omega}\omega - M$, then there is no $g : ({}^{\omega}\omega)^M \to \omega$ in M satisfying

$$(\forall x \in ({}^{\omega}\omega)^M)\, \mathrm{Exit}([[x']])(x) \leq g(x).$$

Recall that we dubbed this encoding $x' \mapsto \mathrm{Exit}([[x']])$ *vertical coding.* One might hope this same trick can be recycled to handle functions from ${}^{\omega}\omega$ to ${}^{\omega}\omega$. We will explain.

Definition VI.13. Given a function $f : {}^{\omega}\omega \to {}^{\omega}\omega$ and $n \in \omega$, the function

$$x \mapsto f(x)(n)$$

from ${}^{\omega}\omega$ to ω is the *n-th slice* of f.

Definition VI.14. Given a sequence $\mathcal{X} = \langle x_n \in {}^{\omega}\omega : n \in \omega \rangle$, $f_{\mathcal{X}} : {}^{\omega}\omega \to {}^{\omega}\omega$ is the function whose n-th slice is $\mathrm{Exit}([[x_n]])$. That is,

$$f_{\mathcal{X}}(x)(n) := \mathrm{Exit}([[x_n]])(x).$$

Suppose M is a transitive model of ZFC and $\mathcal{X} = \langle x_n \in {}^{\omega}\omega : n \in \omega \rangle$ is a sequence such that no x_n is in M. Now, consider an arbitrary $g : ({}^{\omega}\omega)^M \to {}^{\omega}\omega$ in M. Our hope is that by a suitable choice of $\mathcal{X}$, g cannot satisfy

$$(\forall x \in ({}^{\omega}\omega)^M)\, f_{\mathcal{X}}(x) \leq^* g(x). \tag{6.3}$$

For each $n \in \omega$, since $x_n \notin M$, the set

$$X_n := \{x \in ({}^{\omega}\omega)^M : g(x)(n) < f_{\mathcal{X}}(x)(n)\}$$

is non-empty. We see that (6.3) is equivalent to

$$(\forall x \in ({}^{\omega}\omega)^M)\, \{n \in \omega : x \in X_n\} \text{ is finite.}$$

Thus, our hope is for infinitely many X_n to contain a common point. Unfortunately, we cannot ensure this (in ZFC) no matter how cleverly we choose the sequence $\mathcal{X}$. Specifically, if V is a Sacks forcing extension of M, then for *any* sequence $\mathcal{X}$, there is a function $g : ({}^{\omega}\omega)^M \to {}^{\omega}\omega$ in M which satisfies (6.3); this is why we call the encoding scheme *naive* vertical coding. In fact, the function g can be chosen to be Borel with a code in M, and letting $\tilde{g} : {}^{\omega}\omega \to {}^{\omega}\omega$ be the function in V coded by the same Borel code,

$$(\forall x \in {}^{\omega}\omega)\, f_{\mathcal{X}}(x) \leq^* \tilde{g}(x).$$

Establishing this fact is the point of this section. The proof is complicated, so we will warm up with a sequence of easier results which systematically introduce the relevant ideas.

For the rest of this section, let M denote a transitive model of ZFC. First, note that if the sequence

$$\mathcal{X} = \langle x_n \in {}^{\omega}\omega : n \in \omega \rangle$$

is in M, then $f_{\mathcal{X}} \restriction M \in M$, and (6.3) holds when we set $g := f_{\mathcal{X}} \restriction M$. Even if $(\forall n \in \omega)\, x_n \in M$, it does not follow that $\mathcal{X} \in M$. Also, it might be the case that

$$\{n \in \omega : x_n \in M\}$$

is not in M. Despite these last two facts, the situation the reader should imagine is when $(\forall n \in \omega)\, x_n \notin M$ (which of course implies $\mathcal{X} \notin M$). Later, we shall see that the situation becomes further complicated when

$$\{\langle n_1, n_2 \rangle : x_{n_1} = x_{n_2}\}$$

is not in M.

Note that if all the x_n's are the same, then (6.3) is satisfied by the function $g(x) := (n \mapsto n)$, because $(\forall x \in {}^{\omega}\omega)\, f(x) : \omega \to \omega$ is a constant function. This phenomenon can occur even if we require the x_n to all be distinct from one another:

Proposition VI.15. *Suppose V is ${}^{\omega}\omega$-bounding over M. Let X' be the set of limit points of elements of the sequence $\mathcal{X} = \langle x_n : n \in \omega \rangle$. If X' is countable, then there is some $y \in ({}^{\omega}\omega)^M$ satisfying*

$$(\forall x \in {}^{\omega}\omega)\, f_{\mathcal{X}}(x) \leq^* y.$$

Proof. Assuming X' is countable, there is some $y' \in {}^{\omega}\omega$ that eventually dominates each element of

$$\{f_{\mathcal{X}}(x) : x \in X'\}.$$

Since V is ${}^{\omega}\omega$-bounding over M, fix some $y \in ({}^{\omega}\omega)^M$ that eventually dominates both y' and the identity function $n \mapsto n$.

Consider any $x \in {}^{\omega}\omega$. If $x \notin X'$, then there is some neighborhood of x containing only finitely many elements of $\mathcal{X}$, so $f_{\mathcal{X}}(x)$ is bounded by the definition of $f_{\mathcal{X}}$, so of course $f_{\mathcal{X}}(x) \leq^* y$. On the other hand, if $x \in X'$, then

$$f_{\mathcal{X}}(x) \leq^* y' \leq^* y$$

by construction. □

If the set X' in the proposition above is *uncountable*, then by applying the Cantor-Bendixson Theorem to the closed set X', we see that $|X'| = 2^{\omega}$. Indeed, without loss of generality we may assume that the points in $\mathcal{X}$ are dense in ${}^{\omega}\omega$; it does not hurt to add all rational numbers to the sequence $\mathcal{X}$. When we make this assumption, $\mathrm{Im}(f_{\mathcal{X}})$ is unbounded:

Proposition VI.16. *Suppose the set X' of limit points of elements of the sequence $\mathcal{X} = \langle x_n : n \in \omega \rangle$ is dense in ${}^{\omega}\omega$. Then $\mathrm{Im}(f_{\mathcal{X}})$ is unbounded. That is, there is no $y \in {}^{\omega}\omega$ (let alone $y \in ({}^{\omega}\omega)^M$) satisfying*

$$(\forall x \in {}^{\omega}\omega)\, f_{\mathcal{X}}(x) \leq^* y.$$

Proof. Consider any $y \in {}^\omega\omega$. We will construct an $x \in {}^\omega\omega$ satisfying $f_\mathcal{X}(x) \not\leq^* y$. That is, an x satisfying

$$(\exists^\infty n \in \omega)\, f_\mathcal{X}(x)(n) > y(n).$$

To build this x, first let $n_0 = 0$. Let $t_0 \in {}^{<\omega}\omega$ be be a node that is *not* an initial segment of x_{n_0}, but t_0 deviates from x_{n_0} after level $y(n_0)$. Next, let $n_1 > n_0$ be such that t_0 *is* an initial segment of x_{n_1}. Such an n_1 exists because $\{x_n : n \in \omega\}$ is dense and $[t_0]$ is an open set. Let $t_1 \in {}^{<\omega}\omega$ be an extension of t_0 that is *not* an initial segment of x_{n_1}, but which deviates from x_{n_1} after level $y(n_1)$. Continuing like this, we get a sequence

$$t_0 \sqsubseteq t_1 \sqsubseteq t_2 \sqsubseteq \ldots.$$

Let $x := \bigcup_{i \in \omega} t_i$. By construction, $f_\mathcal{X}(x)(n_i) > y(n_i)$ for all $i \in \omega$. Hence, $f_\mathcal{X}(x)(n) > y(n)$ for infinitely many n. $\square$

The fact that $\mathrm{Im}(f_\mathcal{X})$ can be unbounded makes it even more shocking that $f_\mathcal{X}$ is pointwise eventually dominated by some $g \in M$ when V is a Sacks forcing extension of M.

The next proposition illustrates a key idea we will later enhance. For simplicity, the reader may want to first consider the case that the x_n's are distinct.

Proposition VI.17. *Let $\mathcal{X} = \langle x_n : n \in \omega \rangle$. Suppose*

$$\mathcal{T} = \langle T_n : n \in \omega \rangle \in M$$

is a sequence of subtrees of ${}^{<\omega}\omega$ satisfying the following:

1) $(\forall n \in \omega)\, x_n \in [T_n]$.

2) $(\forall n_1, n_2 \in \omega)$ *one of the following holds:*

a) $x_{n_1} = x_{n_2}$*;*

b) $[T_{n_1}] \cap [T_{n_2}] = \emptyset$.

Then there is a Borel function $g : {}^\omega\omega \to {}^\omega\omega$ *that has a Borel code in* M *satisfying*

$$(\forall x \in {}^\omega\omega)\, f_{\mathcal{X}}(x) \leq^* g(x).$$

Proof. Let $g : {}^\omega\omega \to {}^\omega\omega$ be defined by

$$g(x)(n) := \max\{\text{Exit}(T_n)(x), n\}.$$

Certainly g is Borel, with a code in M (because $\mathcal{T} \in M$). The "Exit$(T_n)(x)$" part of the definition is doing most of the work. Specifically, for any $n \in \omega$ and $x \notin [T_n]$,

$$f_{\mathcal{X}}(x)(n) = \text{Exit}([[x_n]])(x) \leq \text{Exit}(T_n)(x).$$

This is because since x_n is a path through the tree T_n, $x \notin [T_n]$ implies the level where x exits T_n is not before the level where x differs from x_n. Thus, we have

$$(\forall n \in \omega)\, x \notin [T_n] \Rightarrow f_{\mathcal{X}}(x)(n) \leq g(x)(n).$$

Suppose, towards a contradiction, that there is some $x \in {}^\omega\omega$ satisfying $f_{\mathcal{X}}(x) \not\leq^* g(x)$. Fix such an x. Let A be the infinite set

$$A := \{n \in \omega : f_{\mathcal{X}}(x)(n) > g(x)(n)\}.$$

It must be that $x \in [T_n]$ for each $n \in A$. By hypothesis, this implies $x_{n_1} = x_{n_2}$ for all $n_1, n_2 \in A$. Thus, $f_{\mathcal{X}}(x)(n)$ is the same constant for all $n \in A$. This is a contradiction, because $g(x)(n) \geq n$ for all n. □

In the proposition above, we may think that the sequence $\mathcal{T}$ witnesses that distinct elements of $\mathcal{X}$ are indeed distinct. Said another way, $\mathcal{T}$ is a tool to separate the x_n's. Unfortunately, if

$$\{\langle n_1, n_2 \rangle : x_{n_1} = x_{n_2}\} \notin M,$$

then there can be no such $\mathcal{T} \in M$. Hence, we must enhance the proposition to make further progress.

The next definition is a more complicated analogue of the sequence $\mathcal{T}$ designed to witness the separation of the elements of $\mathcal{X}$ from one another. When a *separation device* $\mathcal{D}$ for $\mathcal{X}$ exists in a transitive model of ZFC, that model can produce a Borel function $g : {}^{\omega}\omega \to {}^{\omega}\omega$ which pointwise eventually dominates $f_{\mathcal{X}}$. However, unlike the case for sequences $\mathcal{T}$ satisfying the hypotheses of the proposition above, it is always the case that M contains a separation device for $\mathcal{X}$ when V is a forcing extension of M by the forcing to add a single Sacks real. This definition was extracted from a longer forcing argument. We present the shorter proof that a separation device exists in the ground model.

In this definition, we fix a canonical bijection $\eta : \omega \to [\omega]^2$ so that for each $\tilde{n} \in \omega$, we may talk about the $\tilde{n}$-th pair $\eta(\tilde{n}) \in [\omega]^2$. That idea is that for each $\{n_1, n_2\} = \eta(\tilde{n}) \in [\omega]^2$, the functions $F_{\tilde{n},n_1}$ and $F_{\tilde{n},n_2}$, together with the finite sets $I(n_1)$ and $I(n_2)$, separate x_{n_1} and x_{n_2} as much as possible. For $n \in \eta(\tilde{n})$, the function $F_{\tilde{n},n} : {}^{\tilde{n}}2 \to \mathcal{P}({}^{<\omega}\omega)$ is shrink-wrapping $2^{\tilde{n}}$ possibilities for the value of x_n. We need to make sure that what contains one possibility for x_{n_1} is sufficiently disjoint from what contains another possibility for x_{n_2}, even if it is not possible that simultaneously both x_{n_1} and x_{n_2} are in the respective containers.

Definition VI.18. A *separation device* $\mathcal{D}$ for $\mathcal{X} = \langle x_n : n \in \omega \rangle$ is a pair $\langle \mathcal{F}, I \rangle$ such that $I : \omega \to [{}^{\omega}\omega]^{<\omega}$ and $\mathcal{F}$ is a collection of functions $F_{\tilde{n},n}$ for $\tilde{n} \in \omega$ and $n \in \eta(\tilde{n})$ which satisfy the following conditions.

1) $F_{\tilde{n},n} : {}^{\tilde{n}}2 \to \mathcal{P}({}^{<\omega}\omega)$ and each element of $\mathrm{Im}(F_{\tilde{n},n})$ is a leafless subtree of ${}^{<\omega}\omega$.

2) $(\exists s \in {}^{\tilde{n}}2)\, x_n \in [F_{\tilde{n},n}(s)]$.

3) given $\{n_1, n_2\} = \eta(\tilde{n})$, $(\forall s_1, s_2 \in {}^{\tilde{n}}2)$ one of the following relationships holds between the sets $C_1 := [F_{\tilde{n},n_1}(s_1)]$ and $C_2 := [F_{\tilde{n},n_2}(s_2)]$:

   3a) $C_1 = C_2$ and if either $x_{n_1} \in C_1$ or $x_{n_2} \in C_2$, then $x_{n_1} = x_{n_2}$;

   3b) $(\exists x \in I(n_1) \cap I(n_2))\, C_1 = C_2 = \{x\}$;

   3c) $C_1 \cap C_2 = \emptyset$, and moreover there is an $l \in \omega$ such that all elements of C_1 deviate from all elements of C_2 before level l.

We do not need all parts of the definition for the next proposition. Specifically, we can replace 3a) with the weaker statement that if $x_{n_2} \in C_2$, then $x_{n_1} = x_{n_2}$. Also, we do not need the function I and we can replace 3b) with the weaker statement that $(\exists x \in {}^{\omega}\omega)\, C_1 = C_2 = \{x\}$. Later, when we show there is always a separation device in the ground model when we perform Sacks forcing, we can easily build the device to satisfy the following additional property for all $\tilde{n} \in \omega$ and $n \in \eta(\tilde{n})$:

4) $(\forall s_1, s_2 \in {}^{\tilde{n}}2)$ one of the following relationships holds between the sets $C_1 := [F_{\tilde{n},n}(s_1)]$ and $C_2 := [F_{\tilde{n},n}(s_2)]$:

   4a) $(\exists x \in I(n))\, C_1 = C_2 = \{x\}$;

   4b) $C_1 \cap C_2 = \emptyset$, and moreover there is an $l \in \omega$ such that all elements of C_1 deviate from all elements of C_2 before level l.

Note this is a requirement on the single function $F_{\tilde{n},n}$ where $n \in \eta(\tilde{n})$, and not a requirement on the pair of functions $\langle F_{\tilde{n},n_1}, F_{\tilde{n},n_2}\rangle$ where $\{n_1, n_2\} = \eta(\tilde{n})$.

Proposition VI.19. *Let $\mathcal{X} = \langle x_n : n \in \omega\rangle$. Suppose*

$$\mathcal{D} = \langle \mathcal{F}, I\rangle \in M$$

is a separation device for $\mathcal{X}$. Then there is a Borel function $g : {}^{\omega}\omega \to {}^{\omega}\omega$ that has a

Borel code in M satisfying

$$(\forall x \in {}^{\omega}\omega)\, f_{\mathcal{X}}(x) \le^* g(x).$$

Proof. For each $n \in \omega$, let $T_n \subseteq {}^{<\omega}\omega$ be the tree

$$T_n := \bigcap\{\bigcup \mathrm{Im}(F_{\tilde{n},n}) : \tilde{n} \in \omega \wedge n \in \eta(\tilde{n})\}.$$

That is, for each $t \in {}^{<\omega}\omega$, $t \in T_n$ iff

$$(\forall \tilde{n} \in \omega)[n \in \eta(\tilde{n}) \Rightarrow t \in \bigcup_{s \in {}^{\tilde{n}}2} F_{\tilde{n},n}(s)].$$

By part 2) of the definition of a separation device,

$$(\forall n \in \omega)\, x_n \in [T_n].$$

Let $e(n_2)$ be the least level l such that if $n_1 < n_2$, $\tilde{n}$ satisfies $\eta(\tilde{n}) = \{n_1, n_2\}$, and $s_1, s_2 \in {}^{\tilde{n}}2$ satisfy $[F_{\tilde{n},n_1}(s_1)] \cap [F_{\tilde{n},n_2}(s_2)] = \emptyset$, then all elements of $[F_{\tilde{n},n_1}(s_1)]$ deviate from all elements of $[F_{\tilde{n},n_2}(s_2)]$ before level l.

Let $g : {}^{\omega}\omega \to {}^{\omega}\omega$ be defined by

$$g(x)(n) := \max\{\mathrm{Exit}(T_n)(x), e(n), n\}.$$

Certainly g is Borel, with a code in M (because $\mathcal{D} \in M$). Just like in the previous proposition, since $x_n \in [T_n]$, for all $x \in {}^{\omega}\omega$ and $n \in \omega$ we have

$$x \notin [T_n] \Rightarrow f_{\mathcal{X}}(x)(n) \le g(x)(n).$$

Suppose, towards a contradiction, that there is some $x \in {}^{\omega}\omega$ satisfying $f_{\mathcal{X}}(x) \not\le^* g(x)$. Fix such an x. Let A be the infinite set

$$A := \{n \in \omega : f_{\mathcal{X}}(x)(n) > g(x)(n)\}.$$

It must be that $x \in [T_n]$ for each $n \in A$. Since A is infinite, we may fix $n_1, n_2 \in A$ satisfying the following:

i) $n_1 < n_2$;

ii) $f_{\mathcal{X}}(x)(n_1) \leq n_2$.

Let $\tilde{n}$ satisfy $\eta(\tilde{n}) = \{n_1, n_2\}$. Since $x \in [T_{n_1}]$, fix some $s_1' \in {}^{\tilde{n}}2$ satisfying

$$x \in [F_{\tilde{n},n_1}(s_1')] =: C_1.$$

Also, since $x_{n_2} \in [T_{n_2}]$, fix some $s_2 \in {}^{\tilde{n}}2$ satisfying

$$x_{n_2} \in [F_{\tilde{n},n_2}(s_2)] =: C_2.$$

By the definition of $e(n_2)$ and the fact that $\text{Exit}([[x_{n_2}]])(x) > e(n_2)$, it cannot be that $C_1 \cap C_2 = \emptyset$. Thus, by part 3) of the definition of a separation device, one of the following holds:

a) $x_{n_1} = x_{n_2}$;

b) $C_1 = C_2 = \{x\}$.

Now, b) cannot be the case because $C_2 = \{x\}$ implies $x_{n_2} = x$, which implies $f_{\mathcal{X}}(x)(n_2) = 0$, which contradicts the fact that $f_{\mathcal{X}}(x)(n_2) > g(x)(n_2)$. On the other hand, a) cannot be the case because $x_{n_1} = x_{n_2}$ implies $f_{\mathcal{X}}(x)(n_1) = f_{\mathcal{X}}(x)(n_2)$, which by ii) implies

$$f_{\mathcal{X}}(x)(n_2) = f_{\mathcal{X}}(x)(n_1) \leq n_2 \leq g(x)(n_2) < f_{\mathcal{X}}(x)(n_2),$$

which is impossible. □

We will soon prove that there is always a separation device in M for a sequence $\mathcal{X}$ when V is a Sacks forcing extension of M. First we present a forcing lemma that is a basic building block for separating x_{n_1} from x_{n_2}. Combining this with a *fusion argument* gives us the result.

Lemma VI.20. *Let $\mathbb{P}$ be any forcing. Let $p_0, p_1 \in \mathbb{P}$ be conditions. Let $\dot{\tau}_0, \dot{\tau}_1$ be names for elements of ${}^\omega\omega$. Suppose that there is no $x \in {}^\omega\omega$ satisfying the following two statements:*

1) $p_0 \Vdash \dot{\tau}_0 = \check{x}$;

2) $p_1 \Vdash \dot{\tau}_1 = \check{x}$.

Then there exist $p_0' \leq p_0$; $p_1' \leq p_1$; and $t_0, t_1 \in {}^{<\omega}\omega$ satisfying the following:

3) $t_0 \perp t_1$,

4) $p_0' \Vdash \dot{\tau}_0 \sqsupseteq \check{t}_0$,

5) $p_1' \Vdash \dot{\tau}_1 \sqsupseteq \check{t}_1$.

Proof. There are two cases to consider. The first is that there exists some $x \in {}^\omega\omega$ such that 1) is true. When this happens, 2) is false. Hence, there exist $t_1 \in {}^{<\omega}\omega$ and $p_1' \leq p_1$ such that 5) is true and $x \perp t_1$. Letting $p_0' := p_0$ and t_0 be some initial segment of x incompatible with t_1, we see that 3) and 4) are true.

The second case is that there is no $x \in {}^\omega\omega$ satisfying 1). When this happens, there exist conditions $p_0^a, p_0^b \leq p_0$ and incompatible nodes $s_a, s_b \in {}^{<\omega}\omega$ satisfying both $p_0^a \Vdash \dot{\tau}_0 \sqsupseteq \check{s}_a$ and $p_0^b \Vdash \dot{\tau}_0 \sqsupseteq \check{s}_b$. Now, it cannot be that both $p_1 \Vdash \dot{\tau}_1 \sqsupseteq \check{s}_a$ and $p_1 \Vdash \dot{\tau}_1 \sqsupseteq \check{s}_b$. Assume, without loss of generality, that $p_1 \nVdash \dot{\tau}_1 \sqsupseteq \check{s}_a$. This implies that there exist $p_1' \leq p_1$ and $t_1 \in {}^{<\omega}\omega$ such that $s_a \perp t_1$ and $p_1' \Vdash \dot{\tau}_1 \sqsupseteq \check{t}_1$. Letting $p_0' := p_0^a$ and $t_0 := s_a$, we are done. $\square$

At this point, the reader may want to think about how to use this lemma to prove that if V is a Sacks forcing extension of M and $\mathcal{X} = \langle x_n : n \in \omega \rangle$ satisfies

$$(\forall n \in \omega)\, x_n \notin M$$

and

$$\{\langle n_1, n_2\rangle : x_{n_1} = x_{n_2}\} \in M,$$

then there is a sequence $\mathcal{T}$ of subtrees of ${}^{\omega}\omega$ satisfying the hypotheses of Proposition VI.17.

The next lemma explains the appearance of I in the definition of a separation device. We are intending the name $\dot{\tau}$ to be such that $\dot{\tau}(n)$ refers to the x_n in the sequence $\mathcal{X} = \langle x_n : n \in \omega\rangle$.

Lemma VI.21. *Consider Sacks forcing $\mathbb{S}$. Let $p \in \mathbb{S}$ be a condition and $\dot{\tau}$ a name satisfying $p \Vdash \dot{\tau} : \omega \to {}^{\omega}\omega$. Then there exists a condition $p' \leq p$ and there exists a function $I : \omega \to [{}^{\omega}\omega]^{<\omega}$ satisfying*

$$p' \Vdash (\forall n \in \omega)\, \dot{\tau}(n) \in \check{V} \to \dot{\tau}(n) \in \check{I}(n).$$

Proof. We may easily construct a function $R : \omega \to \mathbb{S}$ that satisfies the conditions of Lemma C.4 such that $R(\emptyset) \leq p$ and for each $s \in {}^{n}2$, either $R(s) \Vdash \dot{\tau}(n) \notin \check{V}$ or $(\exists x \in {}^{\omega}\omega)\, R(s) \Vdash \dot{\tau}(n) = \check{x}$. Define I as follows:

$$I(n) := \{x \in {}^{\omega}\omega : (\exists s \in {}^{n}2)\, R(s) \Vdash \dot{\tau}(n) = \check{x}\}.$$

Let $p' := \bigcap_n \bigcup\{R(s) : s \in {}^{n}2\}$. The condition p' and the function I are as desired.

□

We are now ready for the main forcing argument of this section.

Proposition VI.22. *Consider Sacks forcing $\mathbb{S}$. Let $p \in \mathbb{S}$ be a condition and $\dot{\tau}$ be a name satisfying $p \Vdash \dot{\tau} : \omega \to {}^{\omega}\omega$. Then there exists a condition $q \leq p$ and there exists a pair $\mathcal{D} = \langle \mathcal{F}, I\rangle$ satisfying*

$$q \Vdash \check{\mathcal{D}} \textit{ is a separation device for } \langle \dot{\tau}(n) : n \in \omega\rangle.$$

Proof. First, let $p' \leq p$ and $I : \omega \to [{}^{\omega}\omega]^{<\omega}$ be given by the lemma above. That is, for each $n \in \omega$,

$$p' \Vdash \dot{\tau}(\check{n}) \in \check{V} \to \dot{\tau}(\check{n}) \in \check{I}(\check{n}).$$

We will define a function $R : {}^{<\omega}2 \to \mathbb{S}$ with $R(\emptyset) \leq p'$ satisfying conditions 1) and 2) of Lemma C.4. At the same time, we will construct a family of functions

$$\mathcal{F} = \langle F_{\tilde{n},n} : \tilde{n} \in \omega, n \in \eta(\tilde{n}) \rangle.$$

Our q will be

$$q := \bigcap_{\tilde{n}} \bigcup_{s \in {}^{\tilde{n}}2} R(s).$$

The function $F_{\tilde{n},n}$ will return a leafless subtree of ${}^{<\omega}\omega$. We will have it so for all $n \in \omega$ and all $\tilde{n}$ satisfying $n \in \eta(\tilde{n})$,

$$(\forall s \in {}^{\tilde{n}}2)\, R(s) \Vdash \dot{\tau}(\check{n}) \in [\check{F}_{\tilde{n},n}(\check{s})].$$

Thus, q will easily force that $\mathcal{D}$ satisfies conditions 1) and 2) of the definition of a separation device. To show that q forces condition 3) of that definition, it suffices to show that for all $\{n_1, n_2\} = \eta(\tilde{n})$ and all $s_1, s_2 \in {}^{\tilde{n}}2$, one of the following holds, where $T_1 := F_{\tilde{n},n_1}(s_1)$ and $T_2 := F_{\tilde{n},n_2}(s_2)$:

3a′) $T_1 = T_2$ and $(\forall s \in {}^{\tilde{n}}2)$,

$$R(s) \Vdash (\dot{\tau}(\check{n}_1) \in [\check{T}_1] \vee \dot{\tau}(\check{n}_2) \in [\check{T}_2]) \to \dot{\tau}(\check{n}_1) = \dot{\tau}(\check{n}_2);$$

3b′) $(\exists x \in I(n_1) \cap I(n_2))\, [T_1] = [T_2] = \{x\}$;

3c′) $[T_1] \cap [T_2] = \emptyset$, and moreover $\mathrm{Stem}(T_1) \perp \mathrm{Stem}(T_2)$.

We will define the functions $F_{\tilde{n},n}$ and the conditions $R(s)$ for $s \in {}^{\tilde{n}}2$ by induction on $\tilde{n}$. Beginning at $\tilde{n} = 0$, let $\{n_1, n_2\} = \eta(0)$. We will define $F_{0,n_1} : {}^{0}2 \to \mathbb{S}$,

$F_{0,n_2} : {}^{0}2 \to \mathbb{S}$, and $R(\emptyset) \le p'$. If $p' \Vdash \dot\tau(\check n_1) = \dot\tau(\check n_2)$, then let $R(\emptyset) := p'$ and define $F_{0,n_1}(\emptyset) = F_{0,n_2}(\emptyset) = T$ where $T \subseteq {}^{<\omega}\omega$ is any leafless tree satisfying $p' \Vdash \dot\tau(\check n_1) \in [\check T]$. This causes 3a′) to be satisfied. If $p' \nVdash \dot\tau(\check n_1) = \dot\tau(\check n_2)$, then let $t_1, t_2 \in {}^{<\omega}\omega$ be incomparable nodes and let $R(\emptyset) \le p'$ satisfy $R(\emptyset) \Vdash \dot\tau(\check n_1) \sqsupseteq \check t_1$ and $R(\emptyset) \Vdash \dot\tau(\check n_2) \sqsupseteq \check t_2$. Then we may define $F_{0,n_1}(\emptyset) = T_1$ and $F_{0,n_2}(\emptyset) = T_2$ where T_1 and T_2 are leafless trees such that $\mathrm{Stem}(T_1) \sqsupseteq t_1$, $\mathrm{Stem}(T_2) \sqsupseteq t_2$, $R(\emptyset) \Vdash \dot\tau(\check n_1) \in [\check T_1]$, and $R(\emptyset) \Vdash \dot\tau(\check n_2) \in [\check T_2]$. This causes 3c′) to be satisfied.

We will now handle the successor step of the induction. Let $\{n_1, n_2\} = \eta(\tilde n)$ for some $\tilde n > 0$. We will define $R(s)$ for each $s \in {}^{\tilde n}2$, and we will define both $F_{\tilde n,n_1}$ and $F_{\tilde n,n_2}$ assuming $R(s')$ has been defined for each $s' \in {}^{<\tilde n}2$. To keep the construction readable, we will start with initial values for the $R(s)$'s and the $F_{\tilde n,n}$'s, and we will modify them as the construction progresses until we arrive at their final values. That is, we will say "replace $R(s)$ with a stronger condition..." and "shrink the tree $F_{\tilde n,n}(s)$...". When we make these replacements, it is understood that still $R(s) \Vdash \dot\tau(\check n) \in [\check F_{\tilde n,n}(\check s)]$. The construction consists of 5 steps.

Step 1: First, for each $s \in {}^{(\tilde n-1)}2$, let $R(s^\frown 0)$ and $R(s^\frown 1)$ be arbitrary extensions of $R(s)$ such that $\mathrm{Stem}(R(s^\frown 0)) \perp \mathrm{Stem}(R(s^\frown 1))$. Also, for each $n \in \{n_1, n_2\}$ and $s \in {}^{\tilde n}2$, let $F_{\tilde n,n}(s)$ be an arbitrary leafless subtree of ${}^{<\omega}\omega$ such that $R(s) \Vdash \dot\tau(\check n) \in [\check F_{\tilde n,n}(\check s)]$.

Step 2: For each $s \in {}^{\tilde n}2$ and $n \in \{n_1, n_2\}$, strengthen $R(s)$ so that either $R(s) \Vdash \dot\tau(\check n) \notin \check V$ or $(\exists x \in I(n))\, R(s) \Vdash \dot\tau(\check n) = \check x$. If the latter case holds, shrink $F_{\tilde n,n}(s)$ so that it has only one path.

Step 3: Fix $n \in \{n_1, n_2\}$. For each pair of distinct $s_1, s_2 \in {}^{\tilde n}2$, strengthen each $R(s_1)$ and $R(s_2)$ and shrink each $F_{\tilde n,n}(s_1)$ and $F_{\tilde n,n}(s_2)$ so that one of the following holds:

i) $(\exists x \in I(n))\, [F_{\tilde{n},n}(s_1)] = [F_{\tilde{n},n}(s_2)] = \{x\}$;

ii) $\mathrm{Stem}(F_{\tilde{n},n}(s_1)) \perp \mathrm{Stem}(F_{\tilde{n},n}(s_2))$.

That is, if i) cannot be satisfied, then we may use Lemma VI.20 to satisfy ii).

Step 4: For each pair of distinct $s_1, s_2 \in {}^{\tilde{n}}2$ such that either $R(s_1) \Vdash \dot{\tau}(\check{n}_1) \notin \check{V}$ or $R(s_2) \Vdash \dot{\tau}(\check{n}_2) \notin \check{V}$, use Lemma VI.20 to strengthen $R(s_1)$ and $R(s_2)$ and shrink $F_{\tilde{n},n_1}(s_1)$ and $F_{\tilde{n},n_1}(s_1)$ so that

$$\mathrm{Stem}(F_{\tilde{n},n_1}(s_1)) \perp \mathrm{Stem}(F_{\tilde{n},n_2}(s_2)).$$

Step 5: For each $s \in {}^{\tilde{n}}2$, do the following: If $R(s) \Vdash \dot{\tau}(\check{n}_1) = \dot{\tau}(\check{n}_2)$, then replace both $F_{\tilde{n},n_1}(s)$ and $F_{\tilde{n},n_2}(s)$ with $F_{\tilde{n},n_1}(s) \cap F_{\tilde{n},n_2}(s)$. Otherwise, strengthen $R(s)$ and shrink $F_{\tilde{n},n_1}(s)$ and $F_{\tilde{n},n_2}(s)$ so that

$$\mathrm{Stem}(F_{\tilde{n},n_1}(s)) \perp \mathrm{Stem}(F_{\tilde{n},n_2}(s)).$$

This completes the construction of $\{R(s) : s \in {}^{\tilde{n}}2\}$, $F_{\tilde{n},n_1}$, and $F_{\tilde{n},n_2}$. We will now prove that it works. Fix $s_1, s_2 \in {}^{\tilde{n}}2$ and let $T_1 := F_{\tilde{n},n_1}(s_1)$ and $T_2 := F_{\tilde{n},n_2}(s_2)$. We must show that one of 3a′), 3b′), or 3c′) holds. The cleanest way to do this is to break into cases depending on whether $s_1 = s_2$ or not.

Case $s_1 \neq s_2$: If either $R(s_1) \Vdash \dot{\tau}(\check{n}_1) \notin \check{V}$ or $R(s_2) \Vdash \dot{\tau}(\check{n}_2) \notin \check{V}$, then by Step 4, we see that 3c′) holds. Otherwise, by Step 2, $(\exists x \in I(n_1))\, [T_1] = \{x\}$ and $(\exists x \in I(n_1))\, [T_2] = \{x\}$. Hence, easily either 3b′) or 3c′) holds.

Case $s_1 = s_2$: If $R(s_1) \not\Vdash \dot{\tau}(\check{n}_1) = \dot{\tau}(\check{n}_2)$, then by Step 5, we see that 3c′) holds. Otherwise, we are in the case that

$$R(s_1) \Vdash \dot{\tau}(\check{n}_1) = \dot{\tau}(\check{n}_2).$$

By Step 5, $T_1 = T_2$. Now, if $R(s_1) \Vdash \dot{\tau}(\check{n}_1) \in \check{V}$, then of course also $R(s_1) \Vdash \dot{\tau}(\check{n}_2) \in \check{V}$, and by Step 2) we see that 3b′) holds. Otherwise, $R(s_1) \Vdash \dot{\tau}(\check{n}_1) \notin \check{V}$. Hence,

$[T_1]$ is not a singleton. We will show that 3a′) holds. Consider any $s \in {}^{\tilde{n}}2$. We must show

$$R(s) \Vdash (\dot{\tau}(\check{n}_1) \in [\check{T}_1] \vee \dot{\tau}(\check{n}_2) \in [\check{T}_1]) \rightarrow \dot{\tau}(\check{n}_1) = \dot{\tau}(\check{n}_2).$$

If $s = s_1$, we are done. Now suppose $s \neq s_1$. It suffices to show

$$R(s) \Vdash \neg(\dot{\tau}(\check{n}_1) \in [\check{T}_1] \vee \dot{\tau}(\check{n}_2) \in [\check{T}_1]).$$

That is, it suffices to show $R(s) \Vdash \dot{\tau}(\check{n}_1) \notin [\check{T}_1]$ and $R(s) \Vdash \dot{\tau}(\check{n}_2) \notin [\check{T}_1]$. Since $s \neq s_1$ and $[T_1]$ is not a singleton, by Step 3, $\mathrm{Stem}(F_{\tilde{n},n}(s)) \perp \mathrm{Stem}(T_1)$. Recall that

$$R(s) \Vdash \dot{\tau}(\check{n}_1) \in [\check{F}_{\tilde{n},n}(\check{s})].$$

Hence, since $[\check{F}_{\tilde{n},n}(\check{s})] \cap [T_1] = \emptyset$, $R(s) \Vdash \dot{\tau}(\check{n}_1) \notin [\check{T}_1]$. By a similar argument, $R(s) \Vdash \dot{\tau}(\check{n}_2) \notin [\check{T}_1]$. This completes the proof. □

We now have the desired result of this section:

Corollary VI.23. *Let $\mathcal{X} = \langle x_n \in {}^{\omega}\omega : n \in \omega \rangle$. Assume V is a forcing extension of M by the forcing to add a single Sacks real. Then there is a Borel function $g : {}^{\omega}\omega \rightarrow {}^{\omega}\omega$ that has a Borel code in M satisfying*

$$(\forall x \in {}^{\omega}\omega)\, f_{\mathcal{X}}(x) \leq^* g(x).$$

Proof. Combine Proposition VI.19 and Proposition VI.22 together. □

A natural question now is which forcings are such that each $f_{\mathcal{X}} : {}^{\omega}\omega \rightarrow {}^{\omega}\omega$ in the extension is pointwise eventually dominated by some function in the ground model. More combinatorially, we may ask about the property that every sequence of reals in the extension has a separation device in the ground model. We have shown that Sacks forcing has this property. It is not obvious whether all ${}^{\omega}\omega$-bounding forcings have this property. It is also not obvious whether the *Sacks property* implies this property.

6.3 Long Projective Well-orderings

In the next chapter, we will prove Theorem VII.28. In the statement of that theorem, it is natural to conjecture that we can remove the requirement that g be Borel and replace it with the weaker requirement that g be projective. This would yield a "Projective Dominator Coding Theorem". Specifically, one could conjecture the following:

Conjecture VI.24. *For each projective function $g : {}^{\omega}\omega \to {}^{\omega}\omega$ there is a countable set $G(g) \subseteq \mathcal{P}(\omega)$ and for each $A \subseteq \omega$ there is a projective function $f_A : {}^{\omega}\omega \to {}^{\omega}\omega$ such that if $g : {}^{\omega}\omega \to {}^{\omega}\omega$ satisfies $(\forall x \in {}^{\omega}\omega)\, f_A(x) \leq^* g(x)$, then $A \in G(g)$.*

What we have in mind for $G(g)$ is the set of elements of $\mathcal{P}(\omega)$ that are definable in the language of set theory using g as a parameter. This conjecture may follow from projective determinacy or large cardinals, which would be very interesting, but there is an obstruction to proving it in ZFC alone. Specifically, the conjecture is false when we assume the following:

1) There is a projective well-ordering of ${}^{\omega}\omega$;

2) $\omega_2 \leq \mathfrak{b}$;

3) The map $(A, x) \mapsto f_A(x)$ is projective.

Let us explain. Statement 2) is equivalent to each subset of ${}^{\omega}\omega$ of size $\leq \omega_1$ being bounded in the poset $\langle {}^{\omega}\omega, \leq^* \rangle$. Statement 3) is satisfied by reasonable encoding schemes (and it is satisfied in Theorem VII.28) There is a model of ZFC which satisfies the first two statements: In [19], Harrington constructs a model in which MA + $\neg$CH holds (and therefore $\mathfrak{b} = 2^{\omega}$) and there is a projective well-ordering of the reals.

Assume 1), 2), and 3). Let $\prec$ be a projective well-ordering of the reals, and let

$$\langle A_\alpha \in \mathcal{P}(\omega) : \alpha < \gamma \rangle$$

be the enumeration of $\mathcal{P}(\omega)$ in the order given by $\prec$. Note that it might be the case that $\gamma > 2^\omega$ (but still $|\gamma| = 2^\omega$). Since $\omega_2 \leq \mathfrak{b}$, for each $x \in {}^\omega\omega$ the set

$$\{f_{A_\alpha}(x) \in {}^\omega\omega : \alpha < \omega_1\}$$

is bounded in the poset $\langle {}^\omega\omega, \leq^* \rangle$. Consider the function $g' : {}^\omega\omega \to {}^\omega\omega$ defined by

$$g'(x) := \text{ the } \prec\text{-least } y \in {}^\omega\omega \text{ such that } (\forall \alpha < \omega_1)\, f_{A_\alpha}(x) \leq^* y.$$

Note that the ordering $\prec$ is used twice in the definition of g'. Since $\prec$ is a projective well-ordering, g' is a projective function. There cannot be a guessing scheme $g \mapsto G(g)$ which accompanies $A \mapsto f_A$ to satisfy the conjecture, because $(\forall \alpha < \omega_1)$

$$(\forall x \in {}^\omega\omega)\, f_{A_\alpha}(x) \leq^* g(x),$$

and it is impossible to guess all of the *uncountably* many sets A_α for $\alpha < \omega_1$ from g using only *countably* many guesses.

6.4 Beyond Pointwise Eventual Domination

The purpose of this section is to provide an upper bound for the type of result in the spirit of Theorem 6.2, which we will prove in the next chapter. Specifically, one might ask the following: for each $A \subseteq \omega$, is there some Borel function $f : {}^\omega\omega \times {}^\omega\omega \to \omega$ such that if $g : {}^\omega\omega \times {}^\omega\omega \to \omega$ is Borel and satisfies

$$(*) \qquad (\forall r \in {}^\omega\omega)(\exists c \in {}^\omega\omega)\, f(r, c) \leq g(r, c),$$

then A is definable from any code for g? That is, the functions are from ${}^\omega\omega \times {}^\omega\omega$ to ω, instead of ${}^\omega\omega \times \omega$ to ω. We will now show that this is *not* a theorem of ZFC. Specifically, we will show that it is false assuming $\neg$CH.

Temporarily let R denote the binary relation defined by fRg iff (*) holds. It suffices to show that there is a size ω_1 family $\mathcal{G}$ of Borel functions from ${}^\omega\omega \times {}^\omega\omega$ to ω such that for each Borel $f : {}^\omega\omega \times {}^\omega\omega \to \omega$, there is some $g \in \mathcal{G}$ satisfying fRg. Combining this with $\neg$CH and assuming towards a contradiction that there is such an encoding scheme $A \mapsto f_A$, by the pigeonhole principle there must be an *uncountable* set $\mathcal{A} \subseteq \mathcal{P}(\omega)$ and a single $g \in \mathcal{G}$ satisfying

$$(\forall A \in \mathcal{A})\, f_A R g.$$

This contradicts the hypothesis on the encoding scheme $A \mapsto f_A$ because for each g, there are only *countably* many $A \in \mathcal{P}(\omega)$ that are definable (in the language of set theory by a formula) using a fixed code for g as a parameter.

The trick is the following easy lemma which allows us to perform a diagonalization:

Lemma VI.25. *For each $\alpha < \omega$, there is a function $g_\alpha : {}^\omega\omega \times {}^\omega\omega \times {}^\omega\omega \to \omega$ whose graph is $\Sigma^0_{\alpha+1}$ such that if $f : {}^\omega\omega \times {}^\omega\omega \to \omega$ is any function whose graph is Σ^0_α, then there is some $a \in {}^\omega\omega$ satisfying*

$$(\forall r, c \in {}^\omega\omega)\, f(r,c) = g_\alpha(a,r,c).$$

Proof. Let $X_\alpha \subseteq {}^\omega\omega \times {}^\omega\omega \times {}^\omega\omega \times \omega$ be a universal Σ^0_α set. That is, X_α is Σ^0_α and for each Σ^0_α set $S \subseteq {}^\omega\omega \times {}^\omega\omega \times \omega$, there is some $a \in {}^\omega\omega$ satisfying

$$(\forall r, c \in {}^\omega\omega)(\forall n \in \omega)[(a,r,c,n) \in X_\alpha \Leftrightarrow (r,c,n) \in S].$$

We will define g_α to be a function whose graph is $\Sigma^0_{\alpha+1}$, were the a-th section of its graph is the same as the a-th section of X_α whenever the latter section is the graph of a function. That is, for each $a, r, c \in {}^\omega\omega$, define $g_\alpha(a, r, c)$ as follows:

$$g_\alpha(a,r,c) := \begin{cases} n & \text{if } (a,r,c,n) \in X_\alpha \wedge (\exists! m)\,(a,r,c,m) \in X_\alpha, \\ 0 & \text{if } \neg(\exists! m)\,(a,r,c,m) \in X_\alpha. \end{cases}$$

This is indeed the graph of a function. Breaking the definition into cases, we see that

$$\begin{aligned} g_\alpha(a,r,c) = n \quad \Leftrightarrow \quad & [n = 0 \wedge (\forall m \in \omega)\,(a,r,c,m) \notin X_\alpha] \vee \\ & [n = 0 \wedge (\exists m_1, m_2 \in \omega)\, m_1 \neq m_2 \wedge \\ & (a,r,c,m_1) \in X_\alpha \wedge (a,r,c,m_2) \in X_\alpha] \vee \\ & [(a,r,c,n) \in X_\alpha \wedge (\forall m \in \omega)\, m \neq n \Rightarrow \\ & (a,r,c,m) \notin X_\alpha]. \end{aligned}$$

Since X_α is Σ^0_α, the graph of g_α is a boolean combination of Σ^0_α sets, so it is $\Sigma^0_{\alpha+1}$. $\square$

Proposition VI.26. *For each $\alpha < \omega_1$, there is a function $g : {}^\omega\omega \times {}^\omega\omega \to \omega$ whose graph is $\Sigma^0_{\alpha+1}$ such that if $f : {}^\omega\omega \times {}^\omega\omega \to \omega$ is a function whose graph is Σ^0_α, then*

$$(\exists a \in {}^\omega\omega)(\forall r \in {}^\omega\omega)\, f(r,a) = g(r,a),$$

so of course

$$(\forall r \in {}^\omega\omega)(\exists c \in {}^\omega\omega)\, f(r,c) \leq g(r,c).$$

Proof. Fix $\alpha < \omega_1$. Let g_α be given by the lemma above. Define $g : {}^\omega\omega \times {}^\omega\omega \to \omega$ by

$$g(r,c) := g_\alpha(c,r,c).$$

Certainly the graph of g is $\Sigma^0_{\alpha+1}$. Now, let $f : {}^\omega\omega \times {}^\omega\omega \to \omega$ be an arbitrary function whose graph is Σ^0_α. By the hypothesis on g_α, there is some $a \in {}^\omega\omega$ satisfying

$$(\forall r, c \in {}^\omega\omega)\, f(r,c) = g_\alpha(a,r,c).$$

Thus,

$$(\forall r \in {}^{\omega}\omega)\, f(r,a) = g_{\alpha}(a,r,a) = g(r,a),$$

and we are done. $\square$

Hence, there is a size ω_1 family $\mathcal{G}$ of Borel functions from ${}^{\omega}\omega \times {}^{\omega}\omega$ to ω such that for each Borel $f : {}^{\omega}\omega \times {}^{\omega}\omega \to \omega$, there is some $g \in \mathcal{G}$ satisfying (*).

CHAPTER VII

Pointwise Eventual Domination Coding Theorems

This chapter is the centerpiece of this thesis, and it contains the deepest results. The encoding techniques we developed to handle functions from ${}^{\omega}\omega$ to ω were a warm-up to handle Borel functions from ${}^{\omega}\omega$ to ${}^{\omega}\omega$. The guiding task will be to prove that $\mathcal{B}_\alpha({}^{\omega}\omega, \leq^*) = 2^\omega$ when $\alpha \geq 1$, but the proofs yield much more.

In the first section, we show how to overcome the problem that the naive vertical encoding scheme had in Section 6.2. The solution to this problem actually gives us the encoding scheme $A \mapsto f_A$ for Theorem VII.28. However, proving that this encoding scheme works is very complicated. We need to perform a forcing-like argument. Section 7.2 is devoted to understanding the poset involved in the argument.

In Section 7.3, we study the situation where $f_A \leq^* g$ and g is a Baire class one function. This is the first non-trivial case of the more general problem where g is Borel. We will construct a morphism from $\mathcal{B}_1({}^{\omega}\omega, \leq^*)$ to $\langle \mathcal{P}(\omega), \leq_{\Delta^1_1} \rangle$. In Section 7.4, we will describe the problems we encounter when g is Baire class two. Getting past this point is a major obstacle. Our approach is to take a step back and understand the abstract purpose of the orderings $\leq$ and $\leq^\star$ introduced in Section 7.2. We will see exactly how we are supposed to use the Prikry-like condition which this pair of orderings satisfies. There is an additional complication which we must endure (the

Ψ function) to get the complexity class Δ^1_2 instead of something larger. Although this is an additional maneuver separate from the other ideas, it drastically affects the structure of the proof.

In Section 7.6 we prove the main theorem: for each $A \subseteq \omega$ and each Borel $g : {}^\omega\omega \to {}^\omega\omega$ which satisfies

$$(\forall x \in {}^\omega\omega)(\exists i \in \omega)\, f_A(x)(i) \leq g(x)(i),$$

A is Δ^1_2 in any code for g. In the final section, we will see that the proof of that theorem yields a rather incredible result: if X and Y are Polish spaces with X uncountable, then for each $A \subseteq \omega$ there is a Borel function $f : X \to Y$ such that whenever $g : X \to Y$ is Borel, one of the following holds:

1) $(\exists x \in X)\, f(x) = g(x)$;

2) A is Δ^1_2 in any code for g.

7.1 Less Naive Coding

In the last chapter, we discovered an obstacle for converting the proof that $\operatorname{cf} \mathcal{B}_{\omega_1}(\omega, \leq) = 2^\omega$ into a proof that $\operatorname{cf} \mathcal{B}_{\omega_1}({}^\omega\omega, \leq^*) = 2^\omega$. Specifically, we showed in Section 6.2 that ZFC cannot prove that given any $a \in {}^\omega\omega$, there exists a sequence of reals $\mathcal{X} = \langle x_n \in {}^\omega\omega : n \in \omega\rangle$ such that whenever $g : {}^\omega\omega \to {}^\omega\omega$ is a Borel function with code c and satisfies $f_{\mathcal{X}} \leq^* g$, then $a \in L[c]$. The problem is that it is consistent (when V is a Sacks forcing extension of an inner model by adding a single real) that every sequence $\mathcal{X}$ of reals can be sufficiently "shrink-wrapped" (using a separation device) without full knowledge of $\mathcal{X}$.

Although no such "naive vertical coding" $a \mapsto f_{\mathcal{X}}$ can work, only a slightly more complicated encoding *will* work. That is, given a sequence of trees

$$\mathcal{T} = \langle T_n \subseteq {}^{<\omega}\omega : n \in \omega \rangle,$$

let $f_{\mathcal{T}} : {}^{\omega}\omega \to {}^{\omega}\omega$ be the function

$$f_{\mathcal{T}}(x)(n) := \text{Exit}(T_n)(x).$$

As a consequence of Theorem VII.28 which we will prove in a few sections, for each real $a \in {}^{\omega}\omega$, there exists a sequence $\mathcal{T}$ of trees satisfying

$$T_0 \subseteq T_1 \subseteq T_2 \subseteq ...$$

such that whenever $g : {}^{\omega}\omega \to {}^{\omega}\omega$ is a Borel function with code c which satisfies $f_{\mathcal{T}} \leq^* g$, then $a \in L[c]$. Let us explain the intuition very informally. The trees should encode the information in a so that anybody who has a real $x \in {}^{\omega}\omega$ but does not know a will have difficulty upper bounding exactly when x exits the tree T_n (if it does at all). It is helpful if we define the trees so that for each $n \in \omega$, the shortest node of x which is not in T_n is still in T_{n+1}. Moreover, the T_n's should somehow "look the same" in the sense that the nodes in $T_{n+1} - T_n$ can be mistaken as nodes in T_n by somebody who does not know a. For example, we do not want all the nodes in $T_{n+1} - T_n$ but none of the nodes in T_n to contain the number 5.

We can give a simple description of the sequence of trees $\mathcal{T}$ we will use in Theorem VII.28. That is, first define from $a \in {}^{\omega}\omega$ any set $A \subseteq \omega$ which codes a. In the proof of that theorem we will build in the additional assumption that A is computable from every infinite subset of itself, but this does not matter here. Then let T_n be the set of all $t \in {}^{<\omega}\omega$ satisfying

$$|\{l \in \text{Dom}(t) : t(l) \in A\}| \leq n.$$

Hence, x exits T_n at the level when x enumerates an element of A for the $(n+1)$-th time. The reader should be convinced that $\mathcal{T}$ satisfies the informal hypotheses we described in the last paragraph.

This next proposition proves that indeed each $a \in {}^\omega\omega$ can be encoded into a sequence of trees $\mathcal{T}$ such that $a \in L[c]$ whenever c is a code for a Borel function $g : {}^\omega\omega \to {}^\omega\omega$ satisfying $f_\mathcal{T} \leq^* g$ and g is of the form

$$g(x) = \max\{\text{Exit}(S_n)(s), y(n)\}$$

for some sequence of trees $\langle S_n \subseteq {}^{<\omega}\omega : n \in \omega \rangle$ and some real $y \in {}^\omega\omega$. Hence, we may overcome the obstruction we discovered in Section 6.2, because the Borel function g we defined there from a separation device was of this form. The reader who trusts us may skip this proof with no loss of continuity. The proof of this proposition uses a different sequence of trees than the one described in the paragraph above to simplify the argument. Also, the trees T_n are subtrees of ${}^{<\omega}3$ instead of ${}^{<\omega}\omega$, which makes the statement slightly stronger. The idea of the proof is for the trees T_n to get bushier and bushier in a homogeneous way.

Proposition VII.1. *For each $a \in {}^\omega 2$, there is a sequence of trees $\mathcal{T} = \langle T_n \subseteq {}^{<\omega}3 : n \in \omega \rangle$ such that whenever $y \in {}^\omega\omega$ and M is a transitive model of ZF which does not contain the real a but does contain a sequence of trees $\langle S_n \subseteq {}^{<\omega}3 : n \in \omega \rangle$ satisfying $(\forall n \in \omega)\, T_n \subseteq S_n$, then there exists an $x \in {}^\omega 3$ satisfying the following for all $n \in \omega$:*

1) $x \in [S_n] - [T_n]$;

2) $y(n) \leq \text{Exit}(T_n)(x)$.

Proof. Let $\langle B_n \subseteq \omega : n \in \omega \rangle$ be a sequence satisfying

- $B_0 = \emptyset$;

- $B_0 \subseteq B_1 \subseteq B_2 \subseteq ...$;
- $(\forall n \in \omega)\, B_{n+1} - B_n$ is infinite.

Certainly, we may choose such a sequence so that it is in every transitive model of ZF. For each $n \in \omega$, B_n will be the set of levels of T_n that are bushy. That is, the numbers in B_n will be the levels of T_n where nodes have exactly 3 children. The other levels will be where nodes of T_n have exactly 2 children. Assume, without loss of generality, that a is computable from each restriction $a \restriction (B_{n+1} - B_n)$. Define T_n to be the unique tree such that $\emptyset \in t$ and for each $t \in T_n$,

$$\mathrm{Succ}_{T_n}(t) = \begin{cases} \{a(|t|), 2\} & \text{if } |t| \notin B_n, \\ \{0, 1, 2\} & \text{if } |t| \in B_n. \end{cases}$$

Notice that

$$T_0 \subseteq T_1 \subseteq T_2 \subseteq ....$$

Now fix a transitive model M of ZF which does not contain $a \in {}^{\omega}2$ but which does contain some fixed sequence of trees $\langle S_n \subseteq {}^{<\omega}3 : n \in \omega \rangle$ satisfying $(\forall n \in \omega)\, T_n \subseteq S_n$. Also fix $y \in {}^{\omega}\omega$. We must build some $x \in {}^{\omega}3$ satisfying 1) and 2) for all $n \in \omega$. Here is the crucial step: by possibly shrinking each S_n, we may assume without loss of generality that

$$S_0 \subseteq S_1 \subseteq S_2 \subseteq ...,$$

and for all $t \in S_n$,

$$\{2\} \subseteq \mathrm{Succ}_{S_n}(t) \subseteq \{0, 1, 2\}$$

and

$$|\mathrm{Succ}_{S_n}(t)| \geq 2.$$

For example, if there was a node $t \in S_0$ satisfying $|\mathrm{Succ}_{S_0}(t)| \leq 1$, then M knows that $t \notin T_0$, so M can remove t from S_0 to get a smaller tree. Now to satisfy 1), we need only have $x \in S_0$ and $(\forall n \in \omega)\, x \notin [T_n]$.

We claim that for each $n \in \omega$ and $t \in S_0$, there exists an extension t' of t in S_0 such that $|t'| \in B_{n+1} - B_n$ and $1 - a(|t'|) \in \mathrm{Succ}_{S_0}(t')$. Moreover, t' can be chosen to be of the form $t^\frown 2^\frown ...^\frown 2$. Suppose, towards a contradiction, that this is not the case. Fix $n \in \omega$ and $t \in S_0$ such that there is no such extension t' of t. Since every element of S_0 has at least two successors, it must be that for each extension t' of t of the form $t^\frown 2^\frown ...^\frown 2$ whose length is in the set $B_{n+1} - B_n$, $\mathrm{Succ}_{S_0}(t') = \{a(|t'|), 2\}$. Hence, for each $i \in \{0, 1\}$ and each $k \in B_{n+1} - B_n$ greater than $|t|$,

$$a(k) = i \Leftrightarrow [\mathrm{Succ}_{S_0}(t^\frown \overbrace{2^\frown ...^\frown 2}^{k-|t|}) = \{i, 2\}].$$

Since we assumed a is computable from $a \restriction (B_{n+1} - B_n)$ and $S_0 \in M$, we have $a \in M$, which is a contradiction. This establishes the claim.

We will now construct an x satisfying 1) and 2). We will inductively construct a sequence $\langle t_n \in {}^{<\omega}\omega : n \in \omega \rangle$ so that

$$t_0 \sqsubseteq t_1 \sqsubseteq t_2 \sqsubseteq ...$$

and for all $n > 0$, $t_n \in S_0 \cap T_n - T_{n-1}$ and $t_n \restriction y(n-1) \in T_{n-1}$. Then $x := \bigcup_{n \in \omega} t_n$ will be as desired.

Let $t_0 := \emptyset$. We will now pick t_1. Of course, $t_0 \in S_0 \cap T_0$. By the claim we showed earlier, there exists an extension t' of t_0 of the form $t_0^\frown 2^\frown ...^\frown 2$ of length at least $y(0)$, such that $|t'| \in B_1 - B_0$ and $1 - a(|t'|) \in \mathrm{Succ}_{S_0}(t)$. Because $t_0 \in S_0 \cap T_0$ and each node in both S_0 and T_0 has a child when concatenating 2, we have that $t' \in S_0 \cap T_0$. Define $t_1 := t'^\frown(1 - a(|t'|))$. By construction $t_1 \in S_0$. The passage from t' to t_1 consists of exiting from T_n but staying within T_{n+1}. That is, since $|t'| \notin B_0$, we have

$t_1 \notin T_0$. Since $|t'| \in B_1$, we have $t_1 \in T_1$. Finally, $t_1 \restriction y(0) \in T_0$, because $|t'| \geq y(0)$ and $t' \in T_0$.

We may now pick t_2 in a similar fashion. We have $t_1 \in S_0 \cap T_1$. By the claim we showed earlier, there exists an extension t' of t_1 of the form $t_1^\frown 2^\frown ... ^\frown 2$ of length at least $y(1)$, such that $|t'| \in B_2 - B_1$ and $1 - a(|t'|)$ is a successor of t' in S_0. Because $t_1 \in S_0 \cap T_1$ and each node in both S_0 and T_1 has a child when concatenating 2, we have that $t' \in S_0 \cap T_1$. Define $t_2 := t^\frown(1 - a(|t|))$. Like before, $t_2 \in S_0 \cap T_2 - T_1$ and $t_1 \restriction y(1) \in T_1$.

We may construct $t_3, t_4, ...$ in the same way, and the proof is complete. $\square$

7.2 Reachability

Within this section, we will present some key concepts needed for Theorem VII.28. We will also use them in Section 7.3 where we warm-up by considering only Baire class one functions. The reader may wish to skip to Section 7.3, returning to this section when needed.

Definition VII.2. Given a set $A \subseteq \omega$ and a pair of nodes $t, t' \in {}^{<\omega}\omega$ such that $t' \sqsupseteq t$, we say that t' *does not hit* A *more than* t if for all $l \in \mathrm{Dom}(t') - \mathrm{Dom}(t)$,

$$t'(l) \notin A.$$

In this situation we write $t' \sqsupseteq^\star t$ (and it should be clear from context what is the set A to which this notation refers).

The intended use for this definition is to facilitate the construction of a real $x \in {}^{\omega}\omega$ as the union of a sequence of nodes $t_0 \sqsubseteq t_1 \sqsubseteq ....$ If $t_{i+k} \sqsupseteq^\star t_i$, then t_{i+k} does not *decide* more of the value $f(x)$ than t_i does. This idea will be clear later.

Definition VII.3. Given a node $t \in {}^{<\omega}\omega$ and a function $h : {}^{<\omega}\omega \to \omega$, a node $t' \in {}^{<\omega}\omega$ is said to be an *extension of* t *to the right of* h, written $t' \sqsupseteq_h t$, if $t' \sqsupseteq t$ and for all $l \in \mathrm{Dom}(t') - \mathrm{Dom}(t)$,

$$t'(l) \geq h(t' \upharpoonright l).$$

We make the similar definition of $x \sqsupseteq_h t$ where $x \in {}^{\omega}\omega$. If both $t' \sqsupseteq_h t$ and $t' \sqsupseteq^{\star} t$ for some fixed set $A \subseteq \omega$, then we write $t' \sqsupseteq_h^{\star} t$.

Definition VII.4. Given $h_1, h_2 : {}^{<\omega}\omega \to \omega$, we write $h_1 \leq h_2$ if

$$(\forall t \in {}^{<\omega}\omega)\, h_1(t) \leq h_2(t).$$

The following is crucial:

Definition VII.5. Given a set $S \subseteq {}^{<\omega}\omega$ and a node $t \in {}^{<\omega}\omega$, we make the following definitions:

- t is *0-S-reachable* if $t \in S$;
- t is *α-S-reachable* for α satisfying $0 < \alpha < \omega_1$ if either t is β-S-reachable for some $\beta < \alpha$, or $\{n \in \omega : t^\frown n \text{ is } \beta\text{-}S\text{-reachable for some } \beta < \alpha\}$ is infinite;
- t is S-reachable if t is α-S-reachable for some $\alpha < \omega_1$. The smallest such α we call $\mathrm{RRank}(t, S)$.

The restriction to countable ordinals is not a loss, because if we extend the definition to all ordinals we would see that each node that is already γ-S-reachable for some ordinal γ is α-S-reachable for some $\alpha < \omega_1$.

Proposition VII.6. *If $t \in {}^{<\omega}\omega$ is not S-reachable, then*

$$(\exists h : {}^{<\omega}\omega \to \omega)(\forall t' \sqsupseteq_h t)\, t' \notin S.$$

Proof. If a node is not S-reachable, then only finitely many of its children are S-reachable. Hence, we can choose $h : {}^{<\omega}\omega \to \omega$ such that $(\forall t' \sqsupseteq_h t)\, t'$ is not S-reachable. In particular, $(\forall t' \sqsupseteq_h t)\, t' \notin S$. □

On the other hand, one can see that if $t \in {}^{<\omega}\omega$ *is* S-reachable, then

$$(\forall h : {}^{<\omega}\omega \to \omega)(\exists t' \sqsupseteq_h t)\, t' \in S.$$

However, in a certain situation, an even stronger statement holds (Proposition VII.9).

Recall that $\omega_1^{CK}(S)$ is the first admissible ordinal over S. That is, the smallest α such that $L_\alpha[S]$ is a model of Kripke-Platek set theory. Equivalently, this is the supremum of the ranks of all well-founded trees recursive in S.

Lemma VII.7. *Given $S \subseteq {}^{<\omega}\omega$, the set of nodes that are S-reachable is Π^1_1 in S. Any node that is S-reachable is α-S-reachable for some $\alpha < \omega_1^{CK}(S)$. Furthermore, given any $\alpha < \omega_1^{CK}(S)$, the set of all nodes that are β-S-reachable for some $\beta < \alpha$ is Δ^1_1 in S.*

Proof. This is an immediate consequence of the theory of inductive and hyperelementary relations as developed in [38]. See also [22] for the theory of inductive definitions. Let $\mathfrak{A}$ be the standard model of arithmetic, with the ability to code elements of ${}^{<\omega}\omega$, adjoined with a unary predicate $\dot{S}$ for the set S. Let R be the set of nodes that are S-reachable. In the language of [38], R is *inductive* on $\mathfrak{A}$. That is, consider the following second-order formula that has a first-order free variable t (to range over $\mathfrak{A}$'s version of ${}^{<\omega}\omega$) and a second-order unary free variable Y:

$$\varphi(t, Y) := t \in \dot{S} \vee t \in Y \vee (\exists^\infty n')\, t^\frown n' \in Y.$$

This is a so-called Y-positive formula because the unary predicate Y occurs positively.

It defines a monotone operator $\Gamma : \mathcal{P}({}^{<\omega}\omega) \to \mathcal{P}({}^{<\omega}\omega)$ by

$$\Gamma(Y) := \{t \in {}^{<\omega}\omega : \varphi(t, Y)\}.$$

For each ordinal α, let

$$R_\alpha := \Gamma(\bigcup_{\beta<\alpha} R_\beta).$$

Note that for each α, R_α is the set of nodes that are α-S-reachable. Let $||\varphi||$ be the smallest ordinal such that $\Gamma(R_{||\varphi||}) = R_{||\varphi||}$. We have $R = R_{||\varphi||}$.

R is the smallest fixed point of Γ, so it is inductive on $\mathfrak{A}$. Hence, R is Π^1_1 on the structure $\mathfrak{A}$, so it is Π^1_1 in S. The *closure ordinal* $\kappa^{\mathfrak{A}}$ of $\mathfrak{A}$ is $\omega_1^{CK}(S)$, so

$$||\varphi|| \leq \kappa^{\mathfrak{A}} = \omega_1^{CK}(S).$$

No element first appears at the $\kappa^{\mathfrak{A}}$-th stage of an inductive definition, so for each $t \in R$ there is some $\alpha < \omega_1^{CK}(S)$ satisfying $t \in R_\alpha$. For any $\alpha < \kappa^{\mathfrak{A}}$, $\bigcup_{\beta<\alpha} R_\beta$ is *hyperelementary* on $\mathfrak{A}$ (both inductive and coinductive on $\mathfrak{A}$) and therefore Δ^1_1 in S. $\square$

It is not hard to find an example of a set $S \subseteq {}^{<\omega}\omega$ such that the set of nodes that are S-reachable is $\Pi^1_1(S)$-complete. As a corollary of the lemma, we have that "being S-reachable is absolute":

Corollary VII.8. *Let M be a transitive model of ZF. Let $t \in {}^{<\omega}\omega$ and $S \subseteq {}^{<\omega}\omega$ be in M. Then (t is S-reachable)^M iff t is S-reachable.*

Proof. This immediately follows from the lemma above and Mostowski's absoluteness theorem. $\square$

This next proposition also uses the lemma above and will be crucial for Lemma VII.22. Technically we can replace Δ^1_1 by Δ^1_2 and the proof of Theorem VII.28 would not be affected (but the proof of Proposition VII.15 would be). However, later we want it to be clear to the reader where Δ^1_2 is coming from. We remind the reader that A is implicit in $\sqsupseteq^\star$.

Proposition VII.9. *Fix $S \subseteq {}^{<\omega}\omega$. If $t \in {}^{<\omega}\omega$ is S-reachable and $A \subseteq \omega$ is a set which is Δ^1_1 in each infinite subset of itself but A is not Δ^1_1 in S, then*

$$(\forall h : {}^{<\omega}\omega \to \omega)(\exists t' \sqsupseteq^\star_h t)\, t' \in S.$$

Proof. Let $\alpha_0 := \mathrm{RRank}(t, S)$. If $\alpha_0 = 0$, then we are done by defining $t' := t$. Otherwise, the set

$$B_0 := \{n : t^\frown n \text{ is } \beta\text{-}S\text{-reachable for some } \beta < \alpha_0\}$$

is infinite. By the lemma above, it is Δ^1_1 in S. The set B'_0 of all elements of B_0 that are $\geq h(t)$ is also infinite and Δ^1_1 in S. It cannot be that $B'_0 \subseteq A$, because if so, then A would be Δ^1_1 in B'_0. By the transitivity of $\leq_{\Delta^1_1}$, we would have that A is Δ^1_1 in S, a contradiction. Thus, fix some $n_0 \in B'_0 - A$.

Next, let $\alpha_1 := \mathrm{RRank}(t^\frown n_0, S)$. If $\alpha_1 = 0$, then we are done by defining $t' := t^\frown n_0$. Otherwise, the set

$$B_1 := \{n : t^\frown n_0^\frown n \text{ is } \beta\text{-}S\text{-reachable for some } \beta < \alpha_1\}$$

is infinite. By the lemma above, it is Δ^1_1 in S. The set B'_1 of all elements of B_1 that are $\geq h(t^\frown n_0)$ is also infinite and Δ^1_1 in S. As before, we may fix some $n_1 \in B'_1 - A$.

We may continue like this. However, the procedure eventually terminates because we are generating a decreasing sequence of ordinals

$$\alpha_0 > \alpha_1 > \ldots. \qquad \square$$

Combining Proposition VII.6 and Proposition VII.9, we get the following crucial fact. One can remember the following slogan: "If we can reach S, then we can star reach S. If we cannot reach S, then we can add a constraint now to prevent us from reaching S later even in a non-star way".

Corollary VII.10 (Reachability Dichotomy). *Fix $t \in {}^{<\omega}\omega$, $A \subseteq \omega$, and $S \subseteq {}^{<\omega}\omega$. If $A \subseteq \omega$ is Δ^1_1 in each infinite subset of itself but A is not Δ^1_1 in S, then exactly one of the following holds:*

1) t is S-reachable, in which case

$$(\forall h : {}^{<\omega}\omega \to \omega)(\exists t' \sqsupseteq^{\star}_{h} t)\, t' \in S;$$

2) t is not S-reachable, in which case

$$(\exists h : {}^{<\omega}\omega \to \omega)(\forall t' \sqsupseteq_{h} t)\, t' \notin S.$$

Frequently, we will have a pair (t, h) with $t \in {}^{<\omega}\omega$ and $h : {}^{<\omega}\omega \to \omega$ and we will need to generate a new pair (t', h') satisfying $t' \sqsupseteq_h t$ (and possibly $t' \sqsupseteq^{\star}_h t$) and $h' \geq h$. The following definition is intended to accommodate this. The reader should think that the orderings are similar to Hechler forcing.

Definition VII.11. Define $\mathbb{H}$ to be the set of pairs (t, h) such that $t \in {}^{<\omega}\omega$ and $h : {}^{<\omega}\omega \to \omega$. We write

$$(t', h') \leq (t, h)$$

if $t' \sqsupseteq_h t$ and $h' \geq h$. We write

$$(t', h') \leq^{\star} (t, h)$$

if $t' \sqsupseteq^{\star}_h t$ and $h' \geq h$.

Corollary VII.10 can now be turned into an abstract statement about the pair of partial orderings $\leq$ and $\leq^\star$:

Observation VII.12. *Fix $A \subseteq \omega$. Let Γ be the set of subsets D of $\mathbb{H}$ of the form $D = \{(t,h) : t \in S\}$ for some $S \subseteq {}^{<\omega}\omega$ such that A is not Δ^1_1 in S. Then for each $D \in \Gamma$ and $p \in \mathbb{H}$, there exists $p' \leq^\star p$ such that either*

$$p' \in D \text{ or } (\forall p'' \leq p')\, p'' \notin D.$$

Note that for an arbitrary poset $\mathbb{P}$ but with two orderings $\leq$ and $\leq^\star$, the statement of the observation above but redefining Γ to be the set of all downward closed subsets D of $\mathbb{P}$ is precisely the *Prikry Condition* ([18]).

7.3 Baire Class One Dominator Coding Theorem

In this section, we will prove that $\operatorname{cf} \mathcal{B}_1({}^\omega\omega, \leq^*) = 2^\omega$. We will do this by constructing a morphism from $\mathcal{B}_1({}^\omega\omega, \leq^*)$ to $\langle \mathcal{P}(\omega), \leq_{\Delta^1_1} \rangle$. Specifically, we will show that for each $A \subseteq \omega$, there is a Baire class one function $f_A : {}^\omega\omega \to {}^\omega\omega$ such that whenever $g : {}^\omega\omega \to {}^\omega\omega$ is Baire class one and satisfies $f_A \leq^* g$, then $A \leq_{\Delta^1_1} c$ where c is any code for g. The function f_A is the same as the one we will use in Theorem VII.28. The function is similar to $f_{\mathcal{T}}$ which we used in in Section 7.1. Let us formally define f_A now in terms of clouds, which will be useful:

Definition VII.13. Fix $A \subseteq \omega$. Given $i \in \omega$, let $C_{A,i} \subseteq {}^{<\omega}\omega$ be the cloud that is the set of all $t \in {}^{<\omega}\omega$ satisfying

$$t(|t| - 1) \in A \text{ and } |\{l < |t| - 1 : t(l) \in A\}| = i.$$

Let $f_A : {}^\omega\omega \to {}^\omega\omega$ be the function

$$f_A(x)(i) := \operatorname{Rep}(C_{A,i})(x).$$

That is, $C_{A,i}$ is the set of all nodes t that enumerate elements of A precisely $i+1$ times and the last value of t is in A. In Section 7.1, we saw how to overcome the obstacle we discovered in Section 6.2. Indeed, the function f_A overcomes this obstacle (if the reader is not convinced from our comments in Section 7.1, this current section should remove all doubt).

The mapping $(A, x) \mapsto f_A(x)$ is projective. So, from what we said in Section 6.3, there cannot be a proof in ZFC that when $g : {}^\omega\omega \rightarrow {}^\omega\omega$ is *any* function satisfying $f_A \leq^* g$, then A is in some countable set associated with g. This is because consistently we may have simultaneously $\omega_2 \leq \mathfrak{b}$ and a projective well-ordering of ${}^\omega\omega$. Thus, in this section we must somehow use the hypothesis that g is Baire class one. We will now explain how.

Suppose g is Baire class one. Each function $x \mapsto g(x)(i)$ is also Baire class one. Hence, by Section 5.1 there exists a sequence of clouds $\langle B_i \subseteq {}^{<\omega}\omega : i \in \omega \rangle$ such that for each $i \in \omega$,

$$(\forall x \in {}^\omega\omega)\, g(x)(i) \leq \mathrm{Rep}(B_i)(x).$$

Such a sequence can be obtained in a canonical way from any code for g. Now suppose A is not Δ^1_1 in a fixed code for g. From the code, we may fix a sequence $\langle B_i : i \in \omega \rangle$ described above such that A is not Δ^1_1 in any B_i. We will use this hypothesis many times to construct an $x \in {}^\omega\omega$ satisfying $(\forall i \in \omega)\, f_A(x)(i) > g(x)(i)$. Indeed, the hypothesis is used many times in Proposition VII.9, and we will use that proposition many times.

We will construct a sequence of nodes

$$t_0 \sqsubseteq t_1 \sqsubseteq t_2 \sqsubseteq ...$$

and our final x will be $\bigcup_i t_i$. We will have each $t_i \in C_{A,i}$. The basic idea is to extend each t_{i-1} to t_i by first hitting B_i *as much as possible* without hitting $C_{A,i}$, and then when we cannot hit B_i any more, we hit $C_{A,i}$ and this will give us our t_i. Since B_i is a cloud, we can only hit it finitely many times! Unfortunately, the constraint that we must wait to hit $C_{A,i}$ prevents us from obtaining a node t *all* of whose extensions are not in B. We must instead be content with the weaker condition that t has a cofinite set of children that are not in B_i, and each of those children has a cofinite set of children that are not in B_i, etc. This was the purpose of introducing the notion of extensions to the right of a function ($t' \sqsupseteq_h t$) in Definition VII.3. Thus, the ability to avoid hitting B_i for the remainder of the construction can be turned into the precise statement that there exists an $h_i : {}^{<\omega}\omega \to \omega$ such that whenever we make extensions to the right of h_i, we will not hit B_i. Since given finitely many functions $h_0, ..., h_i$ we can take their maximum, we can simultaneously avoid hitting $B_0, ..., B_i$ for the remainder of the construction. This next lemma encapsulates "hitting B_i as much as possible until we cannot hit B_i any more". It uses what we developed in the last section:

Lemma VII.14. *Let $A \subseteq \omega$ be Δ^1_1 in each infinite subset of itself. Let $B \subseteq {}^{<\omega}\omega$ be a cloud such that A is not Δ^1_1 in B. Then for each $h : {}^{<\omega}\omega \to \omega$ and $t \in {}^{<\omega}\omega$, there is some $t' \sqsupseteq^\star_h t$ and $h' \geq h$ satisfying*

$$(\forall t'' \sqsupseteq_{h'} t')\, t'' \notin B.$$

Proof. Fix appropriate A, B, h, t. Let $t_0 := t$. There are two cases: either t_0 is B-reachable or not. In each case, we apply the reachability dichotomy (Corollary VII.10). If t_0 is not B-reachable, then we may fix $h' \geq h$ such that $(\forall t'' \geq_{h'} t_0)\, t'' \notin B$, and we are done. Otherwise, t_0 is B-reachable, and we may pick $t'_0 \sqsupseteq^\star_h t_0$ such that $t'_0 \in B$. Properly extend t'_0 to some $t_1 \sqsupseteq^\star_h t'_0$ (so $t_1 \neq t'_0$).

We may continue and again there are two cases: either t_1 is B-reachable or not. If t_1 is not B-reachable, then we may fix $h' \geq h$ such that $(\forall t'' \geq_{h'} t_1)\, t'' \notin B$, and we are done. Otherwise, t_1 is B-reachable, and we may pick $t_1' \sqsupseteq_h^\star t_1$ such that $t_1' \in B$. Properly extend t_1' to some $t_2 \sqsupseteq_h^\star t_1'$. Again, we may again break into cases.

We claim that this procedure eventually terminates. If not, then we have an infinite sequence

$$t_0' \sqsubseteq t_1' \sqsubseteq t_2' \sqsubseteq ...$$

of distinct nodes, all in B. This contradicts B being a cloud. $\square$

We may now present the main result of this section. It uses the function f_A in Definition VII.13.

Proposition VII.15. *For each $A \subseteq \omega$, whenever $g : {}^\omega\omega \to {}^\omega\omega$ is a Baire class one function satisfying*

$$(\forall x \in {}^\omega\omega)(\exists i \in \omega)\, f_A(x)(i) \leq g(x)(i),$$

then A is Δ^1_1 in any code for g.

Proof. Without loss of generality, assume that A is Δ^1_1 in each infinite subset of itself. Indeed, it is straightforward to show that each A is Turing equivalent to a set B which is computable from every infinite subset of itself. Let $g : {}^\omega\omega \to {}^\omega\omega$ be Baire class one. Assume that A is not Δ^1_1 in g. There exists a sequence of clouds $\langle B_i \subseteq {}^{<\omega}\omega : i \in \omega \rangle$ such that for each $i \in \omega$,

$$(\forall x \in {}^\omega\omega)\, g(x)(i) \leq \text{Rep}(B_i)(x)$$

and A is not Δ^1_1 in B_i. The fact that A is not Δ^1_1 in B_i follows from that fact that from any code for g, we may form the clouds B_i in a canonical and simple way (by the theory developed in Section 5.1). We will now define a sequence of nodes

$t_0 \sqsubseteq t_1 \sqsubseteq t_2 \sqsubseteq \ldots$ such that $x := \bigcup_i t_i$ satisfies

$$(\forall x \in {}^\omega\omega)(\forall i \in \omega)\, g(x)(i) < f_A(x)(i).$$

First, use the lemma above with $B := B_0$, h the zero function, and $t := \emptyset$ to obtain $t_0 \sqsupseteq_h^\star \emptyset$ and h_0 satisfying $(\forall t'' \sqsupseteq_{h_0} t_0)\, t'' \notin B_0$. Extend t_0 by one step $t_0' \sqsupseteq_{h_0} t_0$ so that $t_0' \in C_{A,0}$. Of course, if $x \in {}^\omega\omega$ and $x \sqsupseteq t_0'$, then $f_A(x)(0) = |t_0'|$. On the other hand, if $x \in {}^\omega\omega$ and $x \sqsupseteq_{h_0} t_0'$, then $g(x)(0) < |t_0'|$. Thus, as long as we only make extensions of t_0' to the right of h_0, we will have that $g(x)(0) < f_A(x)(0)$.

Next, use the lemma above again with $B := B_1$, $h := h_0$, and $t := t_0'$ to obtain $t_1 \sqsupseteq_{h_0}^\star t_0'$ and $h_1 \geq h_0$ satisfying $(\forall t'' \sqsupseteq_{h_1} t_1)\, t'' \notin B_1$. Extend t_1 by one step $t_1' \sqsupseteq_{h_1} t_1$ so that $t_1' \in C_{A,1}$. By similar reasons to those before, as long as we only make extensions of t_1' to the right of h_1, we will have that $g(x)(1) < f_A(x)(1)$. Continuing like this, our x is as desired. $\square$

We now have the promised morphism:

$$\begin{array}{ccc} \mathcal{B}_1({}^\omega\omega) & \leq^* & \mathcal{B}_1({}^\omega\omega) \\ \uparrow & \Downarrow & \downarrow \\ \mathcal{P}(\omega) & \leq_{\Delta^1_1} & \mathcal{P}(\omega). \end{array}$$

Our next task is to find a morphism from $\mathcal{B}_2({}^\omega\omega, \leq^*)$ to a poset similar to $\langle \mathcal{P}(\omega), \leq_{\Delta^1_1} \rangle$. It will become clear that $\leq_{\Delta^1_1}$ is too restrictive, and we will instead use $\leq_{\Delta^1_2}$.

7.4 Working Towards Baire Class Two Dominators

There are several problems we encounter trying to push the argument from the last section to Baire class two dominators $g : {}^\omega\omega \to {}^\omega\omega$. The crucial problem is that given a node $t \in {}^{<\omega}\omega$, there need not exist an extension $t' \sqsupseteq t$ (let alone an extension

$t' \sqsupseteq_h^\star t$ for some h) and an h' satisfying

$$(\exists l \in \omega)(\forall x \sqsupseteq_{h'} t')\, g(x)(0) \leq l.$$

This is true of the Baire class two function $g : {}^\omega\omega \to {}^\omega\omega$ defined by

$$g(x)(i) := \begin{cases} \max\{x(l) : l < \omega\} & \text{if } \{x(l) : l < \omega\} \text{ is bounded,} \\ 0 & \text{otherwise.} \end{cases}$$

Another problem is that Baire class two functions are not in general dominated by functions represented by clouds. We need the appropriate analogue of Lemma VII.14. In that lemma, we hit a cloud as much as possible by making $\sqsupseteq^\star$-extensions until we could not do so anymore. This was done to *stabilize* the behavior of g. There is a more complicated way to accomplish such stabilization, with the advantage that it generalizes to all functions in the Baire hierarchy. Let us explain the technique now for Baire class one functions, which by now we are quite familiar with.

To simplify the discussion, let $g_\emptyset : {}^\omega\omega \to \omega$ be Baire class one. Let $\langle g_{\langle n\rangle} : n \in \omega\rangle$ be a sequence of continuous functions from ${}^\omega\omega$ to ω such that

$$(\forall x \in {}^\omega\omega)\, g_\emptyset(x) = \lim_{n\to\infty} g_{\langle n\rangle}(x). \tag{7.1}$$

For each $n \in \omega$, let $S_n \subseteq {}^{<\omega}\omega$ be a barrier (Definition IV.3) and $\tilde{g}_{\langle n\rangle} : S_n \to \omega$ be a function specifying $g_{\langle n\rangle}$ as in Proposition IV.4. Fix $l \in \omega$, $h : {}^{<\omega}\omega \to \omega$, and $A \subseteq \omega$. We need to make some assumption about the relationship between A and both the sets S_n and the functions $\tilde{g}_{\langle n\rangle}$. The exact assumption is that A should not be Δ^1_1 in any of the sets S' we will define in the next couple paragraphs.

To stabilize $g_\emptyset$ by making $\sqsupseteq_h^\star$-extensions (to ensure that the final value $g_\emptyset(x)$ is either $\leq l$ or $> l$), we do the following. To begin, we start with $t \in {}^{<\omega}\omega$, and $\sqsupseteq_h^\star$-extend it to some $t_{n_0} \in S_{n_0}$ where $n_0 = 0$ (if t is already below an element of S_{n_0}, we do nothing for this first step and set $t_{n_0} := t$). Without loss of generality, assume $\tilde{g}_{\langle n_0\rangle}(t_{n_0}) \leq l$. There are two cases. Either t_{n_0} is S'-reachable or it is not, where

$$S' := \{t \in {}^{<\omega}\omega : (\exists n > n_0)\, t \in S_n \text{ and } \tilde{g}_{\langle n\rangle}(t) > l\}.$$

If t_{n_0} is not S'-reachable, then we may use the assumption that A is not Δ^1_1 in S' to apply the reachability dichotomy (Corollary VII.10) to get $h' \geq h$ satisfying

$$(\forall t' \sqsupseteq_{h'} t_{n_0})\, t' \notin S'.$$

Hence,

$$(\forall x \sqsupseteq_{h'} t_{n_0})(\forall n > n_0)\, g_{\langle n\rangle}(x) \leq l.$$

Since $g_\emptyset$ is the limit of the functions $g_{\langle n\rangle}$, we have

$$(\forall x \sqsupseteq_{h'} t')\, g_\emptyset(x) \leq l.$$

Thus, we have stabilized $g_\emptyset(x)$ to be $\leq l$ and we are done. The other case is that t_{n_0} is S'-reachable. In this case, we may also apply the reachability dichotomy to get $t_{n_1} \sqsupseteq^\star_h t_{n_0}$ where $n_1 > n_0$ and $\tilde{g}_{\langle n_1\rangle}(t_{n_1}) > l$.

There are again two cases: either t_{n_1} is S'-reachable or it is not, where we redefine S' to be

$$S' := \{t \in {}^{<\omega}\omega : (\exists n > n_1)\, t \in S_n \text{ and } \tilde{g}_{\langle n\rangle}(t) \leq l\}.$$

If t_{n_1} is not S'-reachable, then like before we can get $h' \geq h$ satisfying

$$(\forall x \sqsupseteq_{h'} t_{n_1})\, g_\emptyset(x) > l,$$

and we are done. Otherwise, we apply the reachability dichotomy to get $t_{n_2} \sqsupseteq^\star_h t_{n_1}$ where $n_2 > n_1$ and $\tilde{g}_{\langle n_2\rangle}(t_{n_2}) \leq l$.

We claim that the procedure eventually terminates. If it does not, then we have a sequence of nodes

$$t_{n_0} \sqsubseteq t_{n_1} \sqsubseteq t_{n_2} \sqsubseteq \ldots$$

where for each $i \in \omega$, $\tilde{g}_{\langle n_i \rangle}(t_{n_i}) \le l$ if i is even, and $\tilde{g}_{\langle n_i \rangle}(t_{n_i}) > l$ if i is odd. Thus, defining $x := \bigcup_i t_i$, we see that $g_{\langle n_i \rangle}(x) \le l$ if i is even, and $g_{\langle n_i \rangle}(x) > l$ if i is odd. Hence, $\lim_{n \to \infty} g_{\langle n \rangle}(x)$ does not exist, which is a contradiction.

Thus, to get an appropriate analogue of Lemma VII.14, we used (7.1) in place of the hypothesis that clouds have no infinite descending sequences. This maneuver is important for the proof of Theorem VII.28. To give a complete proof that $\mathrm{cf}\,\mathcal{B}_2({}^{\omega}\omega, \le^*) = 2^{\omega}$, we would need to develop much of the machinery of Theorem VII.28. In the next section, we will discuss the abstract role of the orderings $\le$ and $\le^{\star}$. Knowing their roles, and making a few reasonable assumptions, we will be able to reverse engineer exactly how they should be used. We feel this is the best way to describe how to overcome the crucial problem described at the beginning of this section (that there need not exist $t' \sqsupseteq t$ and h' satisfying $(\exists l \in \omega)(\forall x \sqsupseteq_{h'} t')\, g(x)(0) \le l$). We phrase the question as follows: how can we *ensure* that $g(x)(0) \le l$ for some l? In the next section, as much as possible we will discuss $\le$ and $\le^{\star}$ without referring to their definitions (to understand their abstract roles). This will allow us to reverse engineer the definition of *ensure*.

7.5 A High Level View of the Theorem

The purpose of Theorem VII.28 is to encode an arbitrary set $A \subseteq \omega$ into a function $f : {}^{\omega}\omega \to {}^{\omega}\omega$ and then prove the following: if $g : {}^{\omega}\omega \to {}^{\omega}\omega$ is a Borel function such that $\underline{A \text{ is not } \Delta^1_2}$ in some (any) code for g, then

$$(\exists x \in {}^{\omega}\omega)(\forall i \in \omega)\, f(x)(i) > g(x)(i).$$

The theorem heavily uses the requirement that g be Borel. Building the x is the fascinating part. The basic idea is to perform a forcing-like argument. That is, we have conditions describing the building process so far; a condition consists of the initial segment of x together with a promise for how to perform the remainder of the construction. Specifically, a condition is a pair $(t, h) \in \mathbb{H}$ with $t \in {}^{<\omega}\omega$ and $h : {}^{<\omega}\omega \to \omega$. The final x will be the union of all t's in the chain of conditions that we construct.

There are two orderings on the set of conditions. One is the ordinary extension ordering $\leq$. The other is the *direct* extension ordering $\leq^\star$. We will have

$$p_1 \leq^\star p_2 \Rightarrow p_1 \leq p_2.$$

Without knowledge of these orderings the reader might think, in analogy with Prikry forcing, that direct extensions are those which keep t fixed and modify only h. This is **not** the case! Instead, direct extensions are those extensions which do not decide more of the value of $f(x)$. For each condition (t, h) and each i, at most $|t|$ of the values $f(x)(i)$ have been decided. If we do not mind making the entire proof slightly more complicated, then we can arrange so that when we do decide the value of $f(x)(i)$, we can choose any value in ω we want. Indeed, this is precisely what is needed to prove the more general Theorem VII.30. However we prefer simplicity, so we simply decide the value of $f(x)(i)$ to be $|t|$. That is, we decide $f(x)(i)$ to be the value that is the length we have traveled in our journey to build x. This simplicity is a *feature* we get by considering the domination relation instead of something more complicated.

Now, suppose we are at some condition $p = (t, h)$ in the construction and $f(x)(i)$ has been decided. If no matter how we perform the remainder of the construction (following the promises we have made, which are built into the $\leq$ ordering of the conditions) it will happen that $f(x)(i) \leq g(x)(i)$, then we have failed. Thus, when we decide the value of $f(x)(i)$, we must be absolutely sure we can ensure $f(x)(i) > g(x)(i)$. But what do we mean by *ensure*? Indeed, as is first evident when considering

Baire class two functions, "ensure" cannot have the naive meaning that we *decide* $g(x)(i)$ to be some value $< f(x)(i)$. To be clear, we say q decides $g(x)(i) = l$ iff for *every* chain of conditions

$$q = (t_0, h_0) \geq (t_1, h_1) \geq (t_2, h_2) \geq ... \tag{7.2}$$

with $\lim_{i\to\infty} |t_i| = \infty$, we have $g(\bigcup_k t_k)(i) = l$. On the other hand, "q ensures $g(x)(i) = l$" should mean that for every such chain **which can result from performing the remainder of the construction**, $g(\bigcup_k t_k)(i) = l$. This seems *circular* because we have not yet fully described the construction. However, we break away from circularity by viewing the remainder of the construction as a game where Player II is trying to cause the final x to satisfy $g(x)(i) = l$, and Player I is actually the totality of all other parts of the remainder of the construction.

Now, we have a double standard because we will *decide* the value $f(x)(i)$ but we will only *ensure* the value $g(x)(i)$. We do this simply because the theorem does not require us to take the more technical approach of only ensuring the value of $f(x)(i)$. By the recursive nature of the construction, the more technical approach would cause complicated feedback. However, this point deserves careful thought.

Eventually, we will show that every condition directly extends to one which ensures $g(x)(i) = l$ for some l. Once we do this, the final proof will work as follows. Start with the top condition of the poset. Directly extend it to ensure $g(x)(0) = l_0$ for some l_0. Then, extend that condition to decide $f(x)(0)$ to be some value $> l_0$. Then, directly extend that condition to ensure $g(x)(1) = l_1$ for some l_1. Then, extend that condition to decide $f(x)(1)$ to be some value $> l_1$, etc. During this construction, we will need to make interventions (stepping in and making direct extensions) to make each "ensuring" into a reality. When we finish, we will have $(\forall i)\, f(x)(i) > g(x)(i)$. For the rest of this section, fix $i \in \omega$ (to simplify notation).

Let us now try to reverse engineer exactly what must be meant by "ensure", taking on faith that such a notion exists. Let us assume that the condition q ensures $g(x)(i) = l$. In order to make $g(x)(i) = l$ true at the end of the construction, we must almost certainly intervene infinitely often in the construction of the sequence of conditions. These interventions should probably be direct extensions. This is because making a non-direct extension would cause more $f(x)(i)$ values to be decided, which would further constrain our possible actions. Hence, we take a small leap of faith and adopt the paradigm that we make only non-direct extensions when *we are ready*, and not when *we are required* in order to fulfill a previously made promise that $f(x)(j) > g(x)(j)$ for some $j < i$.

With this concession, we have a reasonable guess for the definition of ensure. Namely, the following: p ensures $g(x)(i) = l$ iff Player II has a winning strategy in the game where Player I makes extensions to the current condition (and the first move extends p) and Player II makes direct extensions to the current condition, where Player II wins iff the real $x := \bigcup_k t_k$ resulting from the construction satisfies $g(x)(i) = l$. Call this game $\mathcal{G}^=(p, g, l)$. For a different function $g' : {}^\omega\omega \to {}^\omega\omega$, the game $\mathcal{G}^=(p, g', l)$ has the analogous definition. Let us now fix the definition:

$$p \text{ ensures } g(x)(i) = l \text{ iff II has a w.s. for } \mathcal{G}^=(p, g, l).$$

The fact that conditions need to be directly extended infinitely often is why we label this proof a forcing-like argument, instead of a literal forcing argument. That is, we see no way to incorporate Player II having a winning strategy for the game into the poset of conditions itself. However, the application of Player II's winning strategy for the game $\mathcal{G}^=(p, g, l)$ would be handled by a Rasiowa-Sikorski argument that uses only direct extensions.

Now that we have a reasonable definition for "ensure", we must ask the following:

does every condition directly extend to one which ensures $g(x)(i) = l$ for some l? We say "directly extend" instead of just "extend" because, again, non-direct extensions will cause additional requirements that we do not want to be bothered with. The answer to this question is yes, but the proof is complicated. Since g is Borel, g is either continuous or the pointwise limit of a sequence of Borel functions with strictly smaller rank in the Baire hierarchy. If g is continuous, then it is easy to see that any condition p can be directly extended to some p' which decides $g(x)(i)$, and therefore p' ensures $g(x)(i) = l$ for some l. On the other hand, if g is the pointwise limit of a sequence $\langle g_n : n \in \omega \rangle$ of Borel functions with strictly smaller rank, then we may assume, as an inductive hypothesis, that

$$(\forall n, p')(\exists p'' \leq^\star p')(\exists l) \text{ II has a w.s. for } \mathcal{G}^=(p'', g_n, l). \tag{7.3}$$

Now fix p. We will argue how to directly extend p to ensure $g(x)(i) = l$ for some l (using an important hypothesis on the pair $\leq$,$\leq^\star$). First, extend p to some $p_0 \leq^\star p$ and fix l_0 such that II has a w.s. for $\mathcal{G}^=(p_0, g_0, l_0)$. Fix such a winning strategy. As we perform the remainder of the construction, apply the winning strategy for this game infinitely often. There are now two cases: either there exists $n_1 > 0$, $p_1 \leq^\star p_0$, and $l_1 \neq l_0$ such that II has a w.s. for $\mathcal{G}^=(p_1, g_{n_1}, l_1)$, or there does not. If there does, then fix such n_1, p_1, l_1, as well as a winning strategy for the game. Apply this winning strategy for the remainder of the construction. We may continue and again there are two cases: either there exists $n_2 > n_1$, $p_2 \leq^\star p_1$, and $l_2 \neq l_1$ such that II has a w.s. for $\mathcal{G}^=(p_2, g_{n_2}, l_2)$, or there does not. If there does, then we may continue as before. However, we claim that eventually the other case holds. This is because if not, then since $(\forall k \in \omega)$ we have ensured $g_{n_k}(x)(i) = l_k$, we get that the limit

$$\lim_{k \to \infty} g_{n_k}(x)(i)$$

does not exist, which contradicts that the limit exists and equals $g(x)(i)$. Thus, at some point in the construction, we must have $n_k \in \omega$, $p_k \leq^\star p$, and l_k such that there does *not* exist $n_{k+1} > n_k$, $p_{k+1} \leq^\star p_k$, and $l_{k+1} \neq l_k$ such that II has a w.s. for $\mathcal{G}^=(p_{k+1}, g_{n_{k+1}}, l_{k+1})$. Fix these values n_k, p_k, l_k; we will be here a while.

Thus, by (7.3), for each $n > n_k$ and $p' \leq^\star p_k$, there do exist $p'' \leq^\star p'$ and l such that II has a w.s. for $\mathcal{G}^=(p'', g_n, l)$, and when this happens l must equal l_k. Informally, this can be remembered as "we can ensure $g_n(x)(i)$ for any particular $n > n_k$, and when we do we have no choice but to ensure it to equal l_k". Now, we have a good idea for how to ensure that $g(x)(i) = l$: Player II should use the strategy for $\mathcal{G}^=(p_k, g, l_k)$ where each move consists of the following:

1) First, directly extend the current condition to some p' so that II has a w.s. for $\mathcal{G}^=(p', g_n, l_k)$, where $n > n_k$ is the smallest n for which this has not yet been done. Fix such a winning strategy.

2) Apply one move of the strategy from part 1). Also, apply one move of each strategy that has resulted from applying part 1) in some previous move.

This looks like it works, but actually there is a subtle problem. That is, in order for this strategy to work, we actually need the following strong statement to hold: for every $n > n_k$ and $p' \leq p_k$, there does exist $p'' \leq^\star p'$ and l such that II has a w.s. for $\mathcal{G}^=(p'', g_n, l)$, and when this happens l must equal l_k. The difference between this statement and the one we made before is that $p' \leq p_k$ instead of just $p' \leq^\star p_k$. The danger is that there could be $p' \leq p_k$ such that when we directly extend p' to ensure $g_n(x)(i) = l$ for some l, we actually have $l \neq l_k$. Our informal way of remembering the weaker statement now sounds dishonest. Let us fix the problem.

Consider the set S_k of conditions p' which extend p_k such that there exists $n > n_k$

and $l \neq l_k$ such that II does have a w.s. for $\mathcal{G}^{=}(g_n, p', l)$. We have that no direct extension of p_k is in S_k, and we want some *direct extension* of p_k such that no *extension* of that condition is in S_k. We have not yet used anything specific about the definition of $\leq$ or $\leq^\star$, nor have we used the hypothesis that A is not Δ^1_2 in g. Here is where we use them. The set S_k is not arbitrary, but occurs in some specific complexity class Γ. Indeed, the definition of S_k only involves a small number of real quantifiers and uses the sequence $\langle g_n : n \in \omega \rangle$. The following axiomatic relationship between $\leq$, $\leq^\star$, and Γ is what we want: whenever p is a condition and $S \in \Gamma$ is a set of conditions, then either there is some direct extension of p in S, or there is some direct extension of p all of whose extensions are not in S. We have already observed (Observation VII.12) that our specific definitions of $\leq$ and $\leq^\star$ cause this relationship to hold. By what we have argued in this section, we see that such a relationship is *necessary.*

At this point, we have described a very general method which only uses a simple axiomatic requirement on $\leq$, $\leq^\star$, and some class Γ. We hope that this underlying method will have applications beyond "encoding information into challenges".

Everything we said so far is true, but the set S_k we have defined two paragraphs ago is, in general, more complicated than Δ^1_2. Part of the complexity comes from the poset $\mathbb{H}$ of conditions itself. If the reader does not mind a sloppy result, then what we have said so far in this section, together with the specific definitions of $\leq$ and $\leq^\star$, can be put together into a proof. Instead of Δ^1_2, we have a larger complexity class. To get the sharper result of Δ^1_2, we need to perform a miraculous technical maneuver. On the one hand, the reader should think of this as an extra technicality that sits on top of the core argument we have given. On the other hand, the maneuver affects the structure of the entire argument.

Fix a well-founded tree $U \subseteq {}^{<\omega}\omega$ and for each $u \in U$ fix a Borel function g_u such that g_u is continuous when u is a leaf node of U, and g_u is the pointwise limit of the functions assigned to the children of u when u is a non-leaf node. We will introduce a recursively defined partial function Ψ taking the arguments u, t, l. The recursiveness of the definition is so that the graph of Ψ is Δ^1_2. However, proving the function is well-defined will be done by induction (on u), and this step cannot be isolated from other statements also being proved by induction (on u). The reader should have the intuition that $\Psi(t, u, l) = 1$ implies that $(\exists h : {}^{<\omega}\omega \to \omega)$ II has a w.s. for $\mathcal{G}^{\leq}((t, h), g_u, l)$ and $\Psi(t, u, l) = 0$ implies that $(\exists h : {}^{<\omega}\omega \to \omega)$ II has a w.s. for $\mathcal{G}^{>}((t, h), g_u, l)$ (where $\mathcal{G}^{\leq}$ and $\mathcal{G}^{>}$ are like $\mathcal{G}^{=}$ but with their winning conditions modified to use $\leq$ and $>$ instead of $=$). However, this fact also will be proved by induction (on u), and this cannot be isolated from other statements being proved by induction. To make the induction work, there is a third statement which we need to prove by induction (on u), which again is done simultaneously with the other statements. The statement is that any condition (p, h) can be *directly* extended to some (p', h') such that $\Psi(t', u, l)$ is defined. Once all these statements have been proved, the proof is completed using the approach described in this section.

The function Ψ is rather disconcerting. It is difficult to say precisely what it means. It is defined recursively, and it only means what it means. On the other hand, when it is defined to be a certain value, this implies a coherent statement involving the existence of winning strategies for the Player II's of the games we described.

When we define the Ψ function in the next section, it will take a node t instead of a pair $(t, h) \in \mathbb{H}$ as an argument. We thought this would simplify the presentation, although the argument works equally well the other way, making the appropriate modifications. We have also taken the approach of keeping the induction on the well-founded tree U as simple as possible (by using the games $\mathcal{G}^{\leq}$ and $\mathcal{G}^{>}$ instead of $\mathcal{G}^{=}$). As a side effect we must perform cleanup work afterwards, but the reader should view this as straightforward.

7.6 Borel Dominator Δ^1_2 Coding Theorem

7.6.1 Fixing A, f_A, g, and U

For the remainder of this section until the statement of the theorem, fix a set $A \subseteq \omega$ which is Δ^1_1 in any infinite subset of itself and fix a Borel function $g : {}^\omega\omega \to {}^\omega\omega$ such that A is not Δ^1_2 in a fixed code for g. We will speak of *the* code for g. Such sets A are easy to construct, and every set A' is Turing equivalent to one which is computable from any infinite subset of itself. The proof would still work even if we only required A to be Δ^1_2 in any infinite subset of itself, but this is not important. We will use the (horizontal) encoding function f_A (Definition VII.13).

Since g occurs somewhere in the Baire hierarchy, using the code for g we may fix a well-founded tree $U \subseteq {}^{<\omega}\omega$ as well as a function $g_u : {}^\omega\omega \to {}^\omega\omega$ for each $u \in U$ satisfying the following:

1) If $u \in U$ is a leaf node of U, then g_u is continuous;

2) If $u \in U$ is not a leaf node of U, then

    i) $(\forall n \in \omega)\, u^\frown n \in U$;

    ii) $(\forall i \in \omega)(\forall x \in {}^\omega\omega)\, g_u(x)(i) = \lim_{n\to\infty} g_{u^\frown n}(x)(i)$;

3) $g = g_\emptyset$.

7.6.2 The function Ψ

We will recursively define a partial function Ψ. Let $t \in {}^{<\omega}\omega$, $u \in U$, and $l, i \in \omega$. The reader may want to think that l and i are fixed. We break the definition into cases, depending on whether $u \in U$ is or is not a leaf node of U. If u is a leaf node of U, $t \in {}^{<\omega}\omega$, and $l, i \in \omega$, then define

$$\Psi(t,u,l,i) := \begin{cases} 1 & \text{if } (\forall x \sqsupseteq t)\, g_u(x)(i) \leq l, \\ 0 & \text{if } (\forall x \sqsupseteq t)\, g_u(x)(i) > l, \\ \uparrow & \text{otherwise.} \end{cases}$$

If u is a non-leaf node of U, $l, i, n \in \omega$, and $c \in \{0,1\}$, then define

$$S(u,n,c,l,i) := \{t' \in {}^{<\omega}\omega : (\exists n' \geq n)\, \Psi(t', u^\frown n', l, i) = c\}.$$

If u is a non-leaf node of U, $t \in {}^{<\omega}\omega$, and $l, i \in \omega$, then define

$$\Psi(t,u,l,i) := \begin{cases} 1 & \text{if } (\exists n \in \omega)\, t \text{ is not } S(u,n,0,l,i)\text{-reachable}, \\ 0 & \text{if } (\exists n \in \omega)\, t \text{ is not } S(u,n,1,l,i)\text{-reachable}, \\ \uparrow & \text{otherwise.} \end{cases}$$

Given $c \in \{0,1\}$, the statement $\neg\Psi(t,u,l,i) = c$ is equivalent to

$$\Psi(t,u,l,i)\downarrow \Rightarrow \Psi(t,u,l,i) = 1 - c,$$

so we may write the non-leaf node case of the definition of Ψ as follows:

$$\Psi(t,u,l,i) := \begin{cases} 1 & \text{if } (\exists n \in \omega)(\exists h)(\forall t' \sqsupseteq_h t)(\forall n' \geq n) \\ & \Psi(t', u^\frown n', l, i)\downarrow \Rightarrow \Psi(t', u^\frown n', l, i) = 1, \\ 0 & \text{if } (\exists n \in \omega)(\exists h)(\forall t' \sqsupseteq_h t)(\forall n' \geq n) \\ & \Psi(t', u^\frown n', l, i)\downarrow \Rightarrow \Psi(t', u^\frown n', l, i) = 0, \\ \uparrow & \text{otherwise.} \end{cases}$$

Temporarily fix a non-leaf node u of U. From the definition, it is not clear whether $\Psi(t,u,l,i)$ is well-defined, because perhaps there is some n and h satisfying $(\forall t' \sqsupseteq_h t)(\forall n' \geq n)\, \Psi(t', u^\frown n', l, i)\uparrow$. This is impossible because

$$(\forall n' \in \omega)(\forall h)(\exists t' \sqsupseteq_h t)\, \Psi(t', u^\frown n', l, i)\downarrow .$$

This will be shown by proving the stronger statement

$$(\forall n' \in \omega)(\forall h)(\exists t' \sqsupseteq_h^\star t)\, \Psi(t', u^\frown n', l, i)\downarrow .$$

That is, we will show

$$(\forall n' \in \omega)\, \Phi(u^\frown n', l, i),$$

where Φ will be defined later. Thus, the fact that Ψ is indeed well-defined will be one of the facts we prove inductively (and simultaneously) using the well-founded tree U. These details have been included for completeness, but the reader should not get bogged down by them. To keep the situation straight, the reader may remember the following:

$$[(\forall n' \in \omega)\, \Phi(u^\frown n', l, i)] \Rightarrow [(\forall t)\, \Psi(t,u,l,i) \text{ is well-defined}].$$

The reader should have the following intuition about Ψ: in the proof of the theorem, we will construct a sequence of nodes $t_0 \sqsubseteq t_1 \sqsubseteq ...$ in order to construct $x := \bigcup_k t_k$. If $\Psi(t_k,u,l,i) = 1$ for some $k \in \omega$, then by the way that we will construct the sequence of nodes, $g_u(x)(i) \leq l$. On the other hand, if $\Psi(t_k,u,l,i) = 0$ for some $k \in \omega$, then similarly $g_u(x)(i) > l$.

The following is our method for upper bounding the complexity of the graph of Ψ. The reader who trusts us may skip to Corollary VII.17, whose statement will be important later.

Proposition VII.16. *Assuming that Ψ is well-defined, the graph of Ψ is Δ^1_2 in the code for g.*

Proof. The idea is for trees to witness that the value of $\Psi(t, u, l, i)$ is what it is. These trees must satisfy a Π^1_1 condition which we will describe shortly, and must be well-founded which is another Π^1_1 condition. For notational simplicity, instead of putting all "scratch-work" into the tree itself, we will attach this information to the tree using a function. We will use the following symbols: '0', '1', and '↑'.

Fix l, i. Here is the definition: call a pair (T, F) *good* if two conditions are satisfied. First, T is a tree (a set of elements ordered by a relation $<_T$ closed under initial segments), F is a function with domain T, and for each t, u, l, i there is an element of T of the form (c, t, u, l, i) for some $c \in \{$'1','0','↑'$\}$. Second, the following are satisfied for each $s = (c, t, u, l, i) \in T$:

(1) One of the following holds:

  (a) $c =$'1' and $\Psi(t, u, l, i) = 1$;

  (b) $c =$'0' and $\Psi(t, u, l, i) = 0$;

  (c) $c =$'↑' and $\Psi(t, u, l, i)\uparrow$;

(2) If s is a leaf node of T, then u is a leaf-node of U, $F(s) = \emptyset$, and one of the following holds:

  (a) $c =$'1' and $(\forall x \sqsupseteq t)\, g_u(x)(i) \leq l$;

  (b) $c =$'0' and $(\forall x \sqsupseteq t)\, g_u(x)(i) > l$;

  (c) $c =$'↑' and $(\exists x \sqsupseteq t)\, g_u(x)(i) \leq l$ and $(\exists x \sqsupseteq t)\, g_u(x)(i) > l$;

(3) If s is a non-leaf node of T, then one of the following holds:

(a) $c =$'1' and $F(s)$ is of the form $F(s) = \{h, n\}$ and for all $t' \sqsupseteq_h t$ and $n' \geq n$, there is an immediate successor s' of s in T of the form $s' = (c', t', u^\frown n', l, i)$ for some $c' \in \{$'1','↑'$\}$;

(b) $c =$'0' and $F(s)$ is of the form $F(s) = \{h, n\}$ and for all $t' \sqsupseteq_h t$ and $n' \geq n$, there is an immediate successor s' of s in T of the form $s' = (c', t', u^\frown n', l, i)$ for some $c' \in \{$'0','↑'$\}$;

(c) $c =$'↑', $F(s) = \emptyset$, and **for all** $h : {}^{<\omega}\omega \to \omega$, $n \in \omega$, and $c' \in \{$'0','1'$\}$ there exists $t' \sqsupseteq_h t$ and $n' \geq n$ and an immediate successor s' of s in T of the form $s' = (c', t', u^\frown n', l, i)$.

The real quantifiers in case (2) of the definition are superficial because the function g_u is continuous when u is a leaf-node of U. Note that this is where the code for g is used. However, case (3)(c) of the definition involves a universal real quantifier (which we have written in bold) and this is essential. Thus, the property of a pair (T, F) being good is Π^1_1 in the code for g. Since being well-founded is a Π^1_1 property, the property of (T, F) being good and T being well-founded is Π^1_1 in the code for g.

There are two important facts about good pairs which follow from the fact that Ψ is well-defined. First, for any t, u, l, i, there exists a good pair which witnesses that $\Psi(t, u, l, i)$ is the value that it is, in the sense of case (1) of the definition. Second, any two good pairs will agree on the value of $\Psi(t, u, l, i)$. This allows us to conclude that the graph of Ψ is Δ^1_2 in the code for g.

For example, consider $c = 1$. The statement $\Psi(t, u, l, i) = 1$ is equivalent to saying there exists a good pair (T, F) such that T is well-founded and $(\text{'1'}, t, u, l, i) \in T$, which is a Σ^1_2 statement in the code for g. On the other hand, the statement $\Psi(t, u, l, i) = 1$ is also equivalent to saying that for all good pairs (T, F) with T well-founded, $(\text{'1'}, t, u, l, i) \in T$, which is a Π^1_2 statement in the code for g. □

It is clear that the proposition above can be applied even when we have only shown that Ψ is well-defined for nodes u up to a certain rank in U. That is, for a fixed $u \in U$, if we know that $\Psi(t, u', l, i)$ is well-defined for all t, l, i and all $u' \in U$ extending u, then the proof of the above proposition tells us that

$$\{(t, u', l, i, c) : u' \sqsupseteq u \wedge \Psi(t, u, l, i) = c\}$$

is Δ^1_2 in the code for g. We record this fact in the next corollary, which will be the only result on the complexity of Ψ we need for the remainder of the proof.

Corollary VII.17. *Fix $u \in U$, $n \in \omega$, $c \in \{0,1\}$, and $l, i \in \omega$. Assume $\Psi(t, u', l, i)$ is well-defined for all t and all $u' \in U$ extending u. Then the set $S(u, n, c, l, i)$ is Δ^1_2 in the code for g.*

Proof. Membership in $S(u, n, c, l, i)$ is arithmetical in the graph of Ψ. □

We are now finished with defining Ψ and analyzing its complexity.

7.6.3 The games $\mathcal{G}^{\leq}$, $\mathcal{G}^{>}$, and $\mathcal{G}^{=}$

The function Ψ has an auxiliary role to the games we will now define. That is, what we really care about is Player II having a winning strategy for either $\mathcal{G}^{\leq}$ or $\mathcal{G}^{>}$. However, we need the more technical Ψ function in order to define a statement which will "induct". We will explain this later.

Definition VII.18. Given $t \in {}^{<\omega}\omega$, $h : {}^{<\omega}\omega \to \omega$, $u \in U$, and $l, i \in \omega$, let $\mathcal{G}^{\leq}(t, h, u, l, i)$ be the following infinite two player game: Player I first plays a pair $(t_0, h_0) \leq (t, h)$, then Player II plays a pair $(t_1, h_1) \leq^\star (t_0, h_0)$, then Player I plays a pair $(t_2, h_2) \leq (t_1, h_1)$, etc. That is, Player I plays a pair $\leq$ the current one in the ordering, and Player II plays a pair $\leq^\star$ the current one. The first player who breaks one of these rules automatically loses. Let $x := \bigcup_k t_k$. To avoid trivialities, if x is finite, then Player I wins. If x is infinite, then Player II wins if $g_u(x)(i) \leq l$.

Notice how there is asymmetry in the game $\mathcal{G}^{\leq}(t, h, u, l, i)$ because Player II must play nodes which are $\leq^\star$ extensions of previous conditions. We have an analogous game but with $>$ instead of $\leq$. We also have a game for $=$, which will not be needed for the proof of the main theorem but will be used for the generalization in the next section:

Definition VII.19. Given $t \in {}^{<\omega}\omega$, $h : {}^{<\omega}\omega \to \omega$, $u \in U$, and $l, i \in \omega$, let $\mathcal{G}^{>}(t, h, u, l, i)$ be the game with the same rules as $\mathcal{G}^{\leq}(t, h, u, l, i)$, but with the modified winning conditions: if $x := \bigcup_k t_k$ is infinite, then Player II wins if $g_u(x)(i) > l$. Similarly, $\mathcal{G}^{=}(t, h, u, l, i)$ is the game with the same rules but if x is infinite, then Player II wins if $g_u(x)(i) = l$.

A strategy for Player II for any of these games is a function taking a sequence $\langle (t_0, h_0), ..., (t_k, h_k) \rangle$. Given such a strategy η, we will abuse terminology by saying "apply η to (t_k, h_k)" instead of "apply η to $\langle (t_0, h_0), ..., (t_k, h_k) \rangle$". Really, we need to keep track of the previous moves in the game and give this to Player II. We suppress these bookkeeping details to keep the proof readable. The reader should remember the following: when we ask the Player II of a game to make a move, we tell him which move it is, we tell him all his previous moves, and we tell him that the previous moves of "Player I" are the concatenation of all the construction that occurred between his moves. We are lying to Player II, *because there is no real Player I*: there are only Player II's for other games (that are also being lied to) and an additional special Player I (who will only show up in the body of the proof of the main theorem) and we concatenate their moves together to create a phantom Player I move.

7.6.4 The statement Φ

Because the theorem is proved using a complicated induction, we introduce formal statements to stand for the inductive hypotheses. This will also make the structure of the argument more visible. Given $u \in U$ and $l, i \in \omega$, let $\Phi(u, l, i)$ be the statement

$$\begin{aligned}\Phi(u,l,i) \quad :\Leftrightarrow \quad & (\forall t \in {}^{<\omega}\omega)(\forall h) \\ & (\exists t' \sqsupseteq_h^\star t) \\ & \Psi(t', u, l, i) \downarrow .\end{aligned}$$

Assume u is a non-leaf node of U. Unraveling the definitions, if we assume

$$(\forall n' \in \omega)\, \Phi(u^\frown n', l, i),$$

then $\Phi(u, l, i)$ is equivalent to the statement that for all $(t, h) \in \mathbb{H}$, there exists $t' \sqsupseteq_h^\star t$, $n \in \omega$, and $c \in \{0, 1\}$ such that

$$t' \text{ is not } S(u, n, c, l, i)\text{-reachable.}$$

Let us quickly explain why: The assumption $(\forall n' \in \omega)\, \Phi(u^\frown n', l, i)$ implies that $\Psi(t', u, l, i)$ is well-defined. Then, $\Psi(t', u, l, i) \downarrow$ iff $(\exists c \in \{0, 1\})\, t'$ is not $S(u, n, c, l, i)$-reachable.

7.6.5 The statements $\Xi^{\leq}$ and $\Xi^{>}$ connecting Ψ to $\mathcal{G}^{\leq}$ and $\mathcal{G}^{>}$

We now must connect Ψ to the games. We do this by introducing a formal statement, which also must be proved simultaneously by induction (on u). Let $\Xi^{\leq}(u, l, i)$ be the statement

$$\begin{aligned}\Xi^{\leq}(u,l,i) \quad :\Leftrightarrow \quad & (\forall t \in {}^{<\omega}\omega)[\Psi(t, u, l, i) = 1 \\ & \Rightarrow (\exists h)\, \text{II has a w.s. for } \mathcal{G}^{\leq}(t, h, u, l, i)].\end{aligned}$$

Let $\Xi^{>}(u,l,i)$ be the statement

$$\begin{aligned}\Xi^{>}(u,l,i) \quad :\Leftrightarrow \quad & (\forall t \in {}^{<\omega}\omega)[\Psi(t,u,l,i)=0 \\ & \Rightarrow (\exists h)\,\text{II has a w.s. for } \mathcal{G}^{>}(t,h,u,l,i)].\end{aligned}$$

For fixed $l,i \in \omega$, we will show by induction on the rank of u in U that $\Phi(u,l,i)$, $\Xi^{\leq}(u,l,i)$, and $\Xi^{>}(u,l,i)$ hold. This will take a fair amount of work. Note that for all u,l,i,

$$\begin{aligned}\Phi(u,l,i) \wedge \Xi^{\leq}(u,l,i) \wedge \Xi^{>}(u,l,i) \quad \Rightarrow \quad & (\,\forall(t,h) \in \mathbb{H}\,) \\ & (\,\exists(t',h') \leq^{\star} (t,h)\,)[\\ & \text{II has a w.s. for } \mathcal{G}^{\leq}(t',h',u,l,i) \vee \\ & \text{II has a w.s. for } \mathcal{G}^{>}(t',h',u,l,i)].\end{aligned}$$

For fixed $l,i \in \omega$, one might hope that one can simply show the right hand side of the above implication by induction on u. Indeed, this would be a great simplification, because we would not need to deal with the recursively defined function Ψ. However, such a proof does not work. It appears as if the best way to show that the right hand side holds for all u is to inductively show that the left hand side holds for all u. Isolating the left hand side as the appropriate statement which would "induct" was the main challenge to proving the theorem. Also note that because of the asymmetry in the games $\mathcal{G}^{\leq}$ and $\mathcal{G}^{>}$, it does not follow that if Player II does not have a winning strategy for $\mathcal{G}^{\leq}$, then Player II does have a winning strategy for $\mathcal{G}^{>}$ (and vice versa). This means that we cannot simply invoke Borel determinacy to conclude that either Player II has a winning strategy for $\mathcal{G}^{\leq}$ or Player II has a winning strategy for $\mathcal{G}^{>}$.

7.6.6 The main induction

We now begin the inductive proof, starting at the leaf nodes of U.

Lemma VII.20. *Fix $l, i \in \omega$. Fix $u \in U$, a leaf node of U. Then $\Phi(u, l, i)$.*

Proof. Fix arbitrary $t \in {}^{<\omega}\omega$ and $h : {}^{<\omega}\omega \to \omega$. We will show

$$(\exists t' \sqsupseteq_h^\star t)\, \Psi(t', u, l, i) \downarrow,$$

and the proof will be complete. By the definition of Ψ, it suffices to show

$$(\exists t' \sqsupseteq_h^\star t)(\exists v \in \omega)(\forall x \sqsupseteq t')\, g_u(x)(i) = v.$$

Let $y \in {}^{\omega}\omega$ be such that $y \sqsupseteq_h^\star t$. Since g_u is continuous, there is some $t' \in {}^{<\omega}\omega$ and $v \in \omega$ such that $y \sqsupseteq t' \sqsupseteq t$ and $(\forall x \sqsupseteq t')\, g_u(x)(i) = v$. The $t' \sqsupseteq_h^\star t$ and v are as desired. □

Lemma VII.21. *Fix $l, i \in \omega$. Fix $u \in U$, a leaf node of U. Then $\Xi^{\leq}(u, l, i)$ and $\Xi^{>}(u, l, i)$.*

Proof. We will just show $\Xi^{\leq}(u, l, i)$, as the proof for $\Xi^{>}(u, l, i)$ is similar. Fix an arbitrary $t \in {}^{<\omega}\omega$ such that $\Psi(t, u, l, i) = 1$. Once we show that for some h Player II has a winning strategy for $\mathcal{G}^{\leq}(t, h, u, l, i)$, we will be done. However, by the definition of Ψ for leaf nodes and the definition of $\mathcal{G}^{\leq}(t, h, u, l, i)$, we see that for any h, *any* strategy for Player II (where he ensures that the sequence being constructed is infinite) is a winning strategy! □

We are now ready to handle the non-leaf node case of the inductive proof. We will use three lemmas to show $\Phi(u, l, i), \Xi^{\leq}(u, l, i)$, and $\Xi^{>}(u, l, i)$ respectively.

The next lemma is the heart of the theorem, and it is where we use the facts about reachability and the complexity of Ψ. In fact, it is the only place where we need these results. This makes it a natural bottleneck for the theorem.

Lemma VII.22. *Fix $l, i \in \omega$. Fix $u \in U$, a non-leaf node of U. Assume*

$$(\forall n \in \omega)[\Xi^{\leq}(u^\frown n, l, i) \wedge \Xi^{>}(u^\frown n, l, i)].$$

Also assume that $\Psi(t, u', l, i)$ is well-defined for all t and all $u' \in U$ extending u (including u itself). Then $\Phi(u, l, i)$.

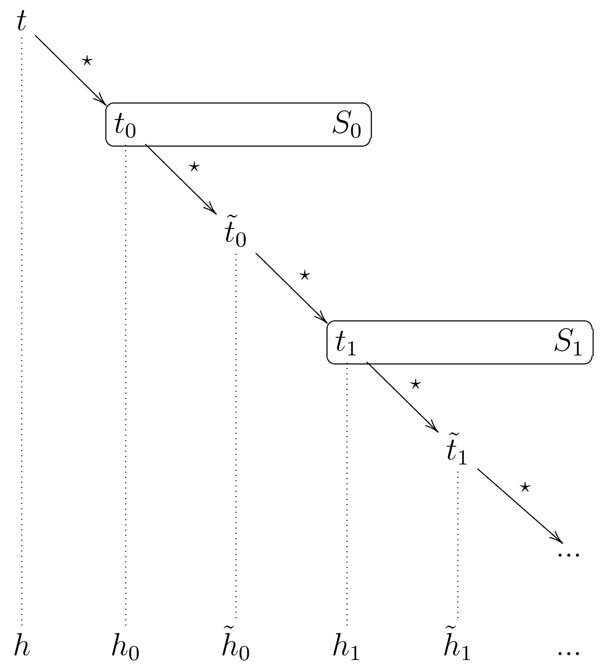
Proof. We will show $\Phi(u, l, i)$. Fix arbitrary $t \in {}^{<\omega}\omega$ and $h : {}^{<\omega}\omega \to \omega$. We will show

$$(\exists t' \sqsupseteq_h^\star t)\, \Psi(t', u, l, i) \downarrow,$$

and the proof will be complete. Since $\Psi(t', u, l, i)$ is well-defined for all t', it suffices to construct $t' \sqsupseteq_h^\star t$, $n \in \omega$, and $c \in \{0, 1\}$ such that

$$t' \text{ is not } S(u, n, c, l, i)\text{-reachable.}$$

Our method of proof is to describe a procedure that we want to terminate in finitely many steps. Assuming the procedure does not terminate, we will reach a contradiction. The reader should use the following diagram to visualize the procedure:



Let $S_0 := S(u, 0, 0, l, i)$. There are two cases. Either t is S_0-reachable or not. If it is not, then we are done by defining $t' := t$, and in this case $\Psi(t', u, l, i) = 1$. Otherwise, t is S_0-reachable, so we proceed as follows:

By Corollary VII.17, the set S_0 is Δ^1_2 in the code for g. Since A is not Δ^1_2 in the code for g and $\leq_{\Delta^1_2}$ is transitive, we have that A is not Δ^1_2 in S_0. This implies that A is not Δ^1_1 in S_0. We may now use Proposition VII.9 to get $t_0 \sqsupseteq^\star_h t$ such that $t_0 \in S_0$. Since $t_0 \in S_0$, fix $n_0 \geq 0$ satisfying

$$\Psi(t_0, u^\frown n_0, l, i) = 0.$$

Since we have assumed $\Xi^>(u^\frown n_0, l, i)$, fix an $h_0 \geq h$ such that Player II has a winning strategy for $\mathcal{G}^>(t_0, h_0, u^\frown n_0, l, i)$. Let η_0 be such a strategy. Note that $(t_0, h_0) \leq^\star (t, h)$. Apply η_0 to the pair (t_0, h_0) to get the pair $(\tilde{t}_0, \tilde{h}_0) \leq^\star (t_0, h_0)$.

Let $S_1 := S(u, n_0, 1, l, i)$. There are two cases. Either $\tilde{t}_0$ is S_1-reachable or not.

If it is not, then we are done by defining $t' := \tilde{t}_0$, and in this case $\Psi(t', u, l, i) = 0$, Otherwise, $\tilde{t}_0$ is S_1-reachable, so we proceed as follows:

As before, A is not Δ^1_2 in S_1, so we may use Proposition VII.9 to get $t_1 \sqsupseteq^\star_{\tilde{h}_0} \tilde{t}_0$ such that $t_1 \in S_1$. Since $t_1 \in S_1$, fix $n_1 > n_0$ satisfying

$$\Psi(t_1, u^\frown n_1, l, i) = 1.$$

Since we have assumed $\Xi^{\leq}(u^\frown n_1, l, i)$, fix an $h_1 \geq \tilde{h}_0$ such that Player II has a winning strategy for $\mathcal{G}^{\leq}(t_1, h_1, u, l, i)$. Let η_1 be such a strategy. Note that $(t_1, h_1) \leq^\star (\tilde{t}_0, \tilde{h}_0)$. Successively apply both η_0 and η_1 (the order does not matter) to the pair (t_1, h_1) to get the pair $(\tilde{t}_1, \tilde{h}_1) \leq^\star (t_1, h_1)$.

We may continue by defining $S_2 := S(u, n_1, 0, l, i)$ and breaking into cases as before. To finish the proof, we will show that this procedure will eventually terminate. Suppose, toward a contradiction, that the procedure goes on forever. This means that we have constructed a sequence of elements of $\mathbb{H}$

$$(t, h) \geq^\star (t_0, h_0) \geq^\star (\tilde{t}_0, \tilde{h}_0) \geq^\star (t_1, h_1) \geq^\star (\tilde{t}_1, \tilde{h}_1) \geq^\star ...,$$

a sequence of numbers

$$n_0 < n_1 < ...,$$

and a sequence of strategies

$$\eta_0, \eta_1, ...$$

such that for each k, η_k is a winning strategy for $\mathcal{G}^{>}(t_k, h_k, u^\frown n_k, l, i)$ if k is even, and η_k is a winning strategy for $\mathcal{G}^{\leq}(t_k, h_k, u^\frown n_k, l, i)$ if k is odd. Let

$$x := \bigcup_k t_k.$$

For each $k \in \omega$, since η_k has been applied infinitely many times in the construction of the sequence of elements of $\mathbb{H}$ and by the rules for the game corresponding to η_k, we see that

$$(\forall k \in \omega) \begin{cases} g_{u^\frown n_k}(x)(i) > l & \text{if } k \text{ is even,} \\ g_{u^\frown n_k}(x)(i) \leq l & \text{if } k \text{ is odd.} \end{cases}$$

This, however, contradicts the fact that $\lim_{n\to\infty} g_{u^\frown n}(x)(i)$ exists. $\square$

The next lemma is much simpler than the previous one. The idea is that to get a winning strategy for Player II of the $\mathcal{G}^{\leq}$ game associated to an internal node $u \in U$, we combine together winning strategies for the Player II's of the $\mathcal{G}^{\leq}$ games associated to the child nodes of u. However, the assumption that $\Psi(t, u, l, i) = 1$ is important.

Lemma VII.23. *Fix $u \in U$, a non-leaf node of U. Fix $l, i \in \omega$. Assume*

$$(\forall n \in \omega)[\Phi(u^\frown n, l, i) \wedge \Xi^{\leq}(u^\frown n, l, i)].$$

Then $\Xi^{\leq}(u, l, i)$.

Proof. Fix arbitrary $t \in {}^{<\omega}\omega$. Assume $\Psi(t, u, l, i) = 1$. Since we are assuming this, fix $p \in \omega$ and h satisfying

$$(\forall t' \sqsupseteq_h t)(\forall n' \geq p)[\Psi(t', u^\frown n', l, i) \downarrow \Rightarrow \Psi(t', u^\frown n', l, i) = 1].$$

We will now describe a winning strategy for Player II for the game $\mathcal{G}^{\leq}(t, h, u, l, i)$, and the proof will be complete.

Let (t_0, h_0) be the first move of Player I. We will describe the first move (t_1, h_1) of Player II. Since $\Phi(u^\frown(p+0), l, i)$, let $t_0' \sqsupseteq_{h_0}^{\star} t_0$ satisfy

$$\Psi(t_0', u^\frown(p+0), l, i) \downarrow .$$

Since $(p+0) \geq p$, we have

$$\Psi(t_0', u^\frown(p+0), l, i) = 1.$$

Since we assumed $\Xi^{\leq}(u^\frown(p+0), l, i)$, fix $h_0' \geq h_0$ and a winning strategy η_{p+0} for Player II for the game

$$\mathcal{G}^{\leq}(t_0', h_0', u^\frown(p+0), l, i).$$

Note that $(t_0', h_0') \leq^\star (t_0, h_0)$. Apply η_0 to the pair (t_0', h_0') to get the pair $(t_1, h_1) \leq^\star (t_0', h_0')$. This concludes Player II's first move.

Now let (t_2, h_2) be the second move of Player I. We will describe the second move (t_3, h_3) of Player II. Since $\Phi(u^\frown(p+1), l, i)$, let $t_2' \sqsupseteq^\star_{h_2} t_2$ be such that

$$\Psi(t_2', u^\frown(p+1), l, i) \downarrow .$$

Since $(p+1) \geq p$, we have

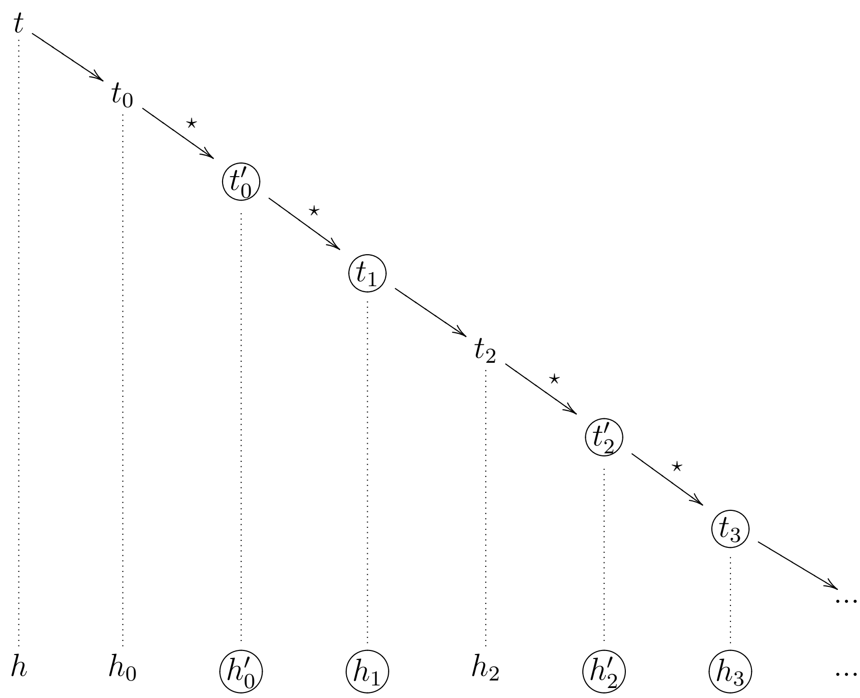
$$\Psi(t_2', u^\frown(p+1), l, i) = 1.$$

Since we assumed $\Xi^{\leq}(u^\frown(p+1), l, i)$, fix $h_2' \geq h_2$ and a winning strategy η_{p+1} for Player II for the game

$$\mathcal{G}^{\leq}(t_2', h_2', u^\frown(p+1), l, i).$$

Note that $(t_2', h_2') \leq^\star (t_2, h_2)$. Successively apply both η_{p+0} and η_{p+1} (the order does not matter) to the pair (t_2', h_2') to get the pair $(t_3, h_3) \leq^\star (t_2', h_2')$. This concludes Player II's second move.

The pattern continues like this. We claim that no matter what moves Player I makes, Player II will win the game $\mathcal{G}^{\leq}(t, h, u, l, i)$ by playing this way. The following diagram helps to visualize the play of the game. The circled entries show which parts of the construction were done by Player II.



Here is why Player II wins: when the game finishes, what has been constructed is a sequence of elements of $\mathbb{H}$

$$(t, h) \geq (t_0, h_0) \geq^\star (t'_0, h'_0) \geq^\star (t_1, h_1) \geq (t_2, h_2) \geq^\star (t'_2, h'_2) \geq^\star (t_3, h_3) \geq ...$$

and a sequence of strategies

$$\eta_{p+0}, \eta_{p+1}, ...$$

such that for each $n \in \omega$, η_{p+n} is a winning strategy for Player II for the game

$$\mathcal{G}^{\leq}(t'_{2n}, h'_{2n}, u^\frown(p+n), l, i).$$

Let

$$x := \bigcup_n t_n.$$

Consider any $n \in \omega$. The strategy η_{p+n} was used infinitely many times for the construction of the sequence of elements of $\mathbb{H}$. All that was done for the construction of that sequence that did not come from the function η_{p+n} can be viewed as the moves of Player I in the game associated to η_{p+n}. Because η_{p+n} is a winning strategy for that game, Player II has won that game, so

$$g_{u^\frown(p+n)}(x)(i) \leq l.$$

Thus, we have shown

$$(\forall n \in \omega)\, g_{u^\frown(p+n)}(x)(i) \leq l.$$

Since

$$g_u(x)(i) = \lim_{n\to\infty} g_{u^\frown n}(x)(i),$$

we have

$$g_u(x)(i) \leq l.$$

That is, Player II has won the game $\mathcal{G}^{\leq}(t, h, u, l, i)$. □

We have an analogous lemma:

Lemma VII.24. *Fix $u \in U$, a non-leaf node of U. Fix $l, i \in \omega$. Assume*

$$(\forall n \in \omega)[\Phi(u^\frown n, l, i) \wedge \Xi^{>}(u^\frown n, l, i)].$$

Then $\Xi^{>}(u, l, i)$.

Proof. The proof is very similar to that of the last lemma, so we will not repeat it. □

Combining the last five lemmas, we immediately have the following:

Corollary VII.25. *For all $u \in U$ and $l, i \in \omega$,*

$$\Phi(u, l, i) \wedge \Xi^{\leq}(u, l, i) \wedge \Xi^{>}(u, l, i).$$

Proof. This is an easy proof by induction on the nodes of the well-founded tree U. Fix $l, i \in \omega$.

Suppose $u \in U$ is a leaf node of U. By Lemma VII.20, $\Phi(u, l, i)$ holds. Hence, for each t, $\Psi(t, u, l, i)$ is well-defined. By Lemma VII.21, both $\Xi^{\leq}(u, l, i)$ and $\Xi^{>}(u, l, i)$ hold.

Suppose $u \in U$ is a non-leaf node of U. Assume that for all $n \in \omega$, $\Phi(u^\frown n, l, i)$, $\Xi^{\leq}(u^\frown n, l, i)$, and $\Xi^{>}(u^\frown n, l, i)$ hold. Also assume that for all t and $u' \in U$ properly extending u, $\Psi(t, u', l, i)$ is well-defined. Since $(\forall n \in \omega)\, \Phi(u^\frown n, l, i)$, for all t we have $\Psi(t, u, l, i)$ is well-defined. By Lemma VII.22, $\Phi(u, l, i)$ holds. By Lemma VII.23, $\Xi^{\leq}(u, l, i)$ holds. By Lemma VII.24, $\Xi^{>}(u, l, i)$ holds. This completes the proof. $\square$

7.6.7 Minor cleanup work

At this point, we are essentially done. The hard work was done in Lemma VII.22, and the corollary above can be used like a black box. However, as a side effect of keeping the hardest part of the proof (the induction on U) simple, we are left with some minor cleanup work. The next two lemmas as well as the theorem in this section and the next should be viewed as easy consequences of the corollary above. The reader may skip this section, trusting us that the lemmas are true when we use them in the theorem.

The next lemma could be proved for arbitrary $u \in U$ instead of just $\emptyset \in U$, but we do not need such generality.

Lemma VII.26. *Fix $i \in \omega$. Assume*

$$(\forall l \in \omega)[\Phi(\emptyset, l, i) \land \Xi^{>}(\emptyset, l, i)].$$

Then

$$(\forall t \in {}^{<\omega}\omega)(\forall h)$$

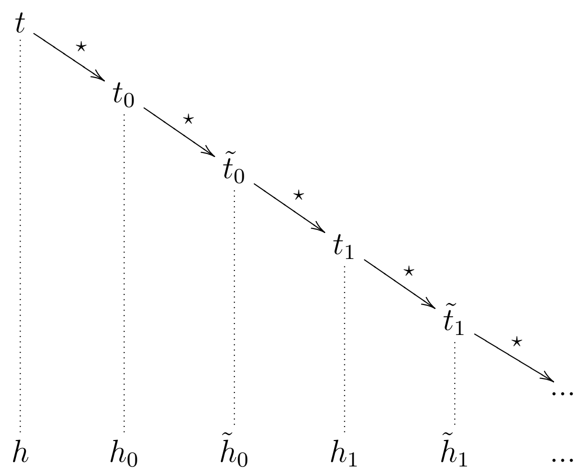
$$(\exists t' \sqsupseteq_h^\star t)(\exists l \in \omega)$$

$$\Psi(t', \emptyset, l, i) = 1.$$

Proof. Fix arbitrary $t \in {}^{<\omega}\omega$ and $h : {}^{<\omega}\omega \to \omega$. We will show

$$(\exists t' \sqsupseteq_h^\star t)(\exists l \in \omega)\, \Psi(t', \emptyset, l, i) = 1,$$

and the proof will be complete. This is another proof where we describe a procedure we want to terminate in finitely many steps. If the procedure goes on forever, then we reach a contradiction. Here is the relevant diagram to guide the reader:



Since $\Phi(\emptyset, 0, i)$ holds, there is some $t_0 \sqsupseteq_h^\star t$ satisfying

$$\Psi(t_0, \emptyset, 0, i)\downarrow.$$

If $\Psi(t_0, \emptyset, 0, i) = 1$, then we are done by defining $t' := t_0$ and $l := 0$. If not, then

$$\Psi(t_0, \emptyset, 0, i) = 0.$$

Since we have assumed $\Xi^>(\emptyset, 0, i)$, fix a function $h_0 \geq h$ and fix a winning strategy η_0 for Player II for the game

$$\mathcal{G}^>(t_0, h_0, \emptyset, 0, i).$$

Note that $(t_0, h_0) \leq^\star (t, h)$. Apply η_0 to the pair (t_0, h_0) to get the pair $(\tilde{t}_0, \tilde{h}_0)$. Note that $(\tilde{t}_0, \tilde{h}_0) \leq^\star (t_0, h_0)$.

Since $\Phi(\emptyset, 1, i)$ holds, there is some $t_1 \sqsupseteq^\star_{\tilde{h}_0} \tilde{t}_0$ satisfying

$$\Psi(t_1, \emptyset, 1, i) \downarrow .$$

If $\Psi(t_1, \emptyset, 1, i) = 1$, then we are done by defining $t' := t_1$ and $l := 1$. If not, then

$$\Psi(t_1, \emptyset, 1, i) = 0.$$

Since we have assumed $\Xi^>(\emptyset, 1, i)$, fix a function $h_1 \geq \tilde{h}_0$ and fix a winning strategy η_1 for Player II for the game

$$\mathcal{G}^>(t_1, h_1, \emptyset, 1, i).$$

Note that $(t_1, h_1) \leq^\star (\tilde{t}_0, \tilde{h}_0)$. Successively apply both η_0 and η_1 (the order does not matter) to the pair (t_1, h_1) to get the pair $(\tilde{t}_1, \tilde{h}_1)$. Note that $(\tilde{t}_1, \tilde{h}_1) \leq^\star (t_1, h_1)$.

The pattern continues like this. We claim that the procedure eventually stops. Suppose, towards a contradiction, that it goes on forever. This means that we have constructed a sequence of elements of $\mathbb{H}$

$$(t, h) \geq^\star (t_0, h_0) \geq^\star (\tilde{t}_0, \tilde{h}_0) \geq^\star (t_1, h_1) \geq^\star (\tilde{t}_1, \tilde{h}_1) \geq^\star ...$$

and a sequence of strategies

$$\eta_0, \eta_1, ...$$

such that for each $l \in \omega$, η_l is a winning strategy for Player II for the game

$$\mathcal{G}^>(t_l, h_l, \emptyset, l, i).$$

Let

$$x := \bigcup_l t_l.$$

Consider any $l \in \omega$. The strategy η_l was used infinitely many times in the construction of the sequence of nodes. All that was done for the construction of the sequence of nodes that did not come from the function η_l can be viewed as the moves of Player I in the game associated with η_l. Because η_l is a winning strategy for that game, Player II has won that game, so

$$g_\emptyset(x)(i) > l.$$

Thus, we have shown

$$(\forall l \in \omega)\, g_\emptyset(x)(i) > l.$$

This is a contradiction. □

This next lemma is not needed for the proof of the main theorem, but it will be used for the generalization in the next section.

Lemma VII.27. *Fix $i \in \omega$. Assume*

$$(\forall l \in \omega)[\Phi(\emptyset, l, i) \wedge \Xi^{\leq}(\emptyset, l, i) \wedge \Xi^{>}(\emptyset, l, i)].$$

Then

$$\begin{aligned}
&(\,\forall (t,h) \in \mathbb{H}\,)\\
&(\,\exists (t',h') \leq^{\star} (t,h)\,)(\exists l \in \omega)\\
&\textit{Player II has a w.s. for } \mathcal{G}^{=}(t',h',\emptyset,l,i).
\end{aligned}$$

Proof. First, use Lemma VII.26 and the fact that $\Xi^{\leq}(\emptyset, l, i)$ holds to get $(t_0, h_0) \leq^{\star} (t, h)$ and $l_0 \in \omega$ such that Player II has a winning strategy η_0 for $\mathcal{G}^{\leq}(t_0, h_0, \emptyset, l_0, i)$. If $l_0 = 0$, we are done by setting $t' := t_0$, $h' := h_0$, and $l := l_0$.

If not, then let $l_1 := l_0 - 1$. Applying $\Phi(\emptyset, l_1, i)$ followed be either $\Xi^{\leq}(\emptyset, l_1, i)$ or $\Xi^{>}(\emptyset, l_1, i)$, we get $(t_1, h_1) \leq^{\star} (t_0, h_0)$ such that Player II has a winning strategy

η_1 for either $\mathcal{G}^{\leq}(t_1, h_1, \emptyset, l_1, i)$ or $\mathcal{G}^{>}(t_1, h_1, \emptyset, l_1, i)$. If η_1 is a winning strategy for $\mathcal{G}^{>}(t_1, h_1, \emptyset, l_1, i)$, then by combining the strategies η_0 and η_1 into one, we have a winning strategy for $\mathcal{G}^{=}(t_0, h_0, \emptyset, l_0, i)$. We are done by setting $t' := t_1$, $h' := h_1$, and $l := l_0$.

Otherwise, η_1 is a winning strategy for $\mathcal{G}^{\leq}(t_1, h_1, \emptyset, l_1, i)$. We may inductively continue the process now starting at l_1 until it eventually stops (in a finite number of steps). □

An alternative induction for proving the main lemmas would have involved proving the generalization of the last lemma to an arbitrary $u \in U$, but we believe the current proof is simpler. That is, we chose to keep the induction on U simple.

7.6.8 Proof of theorem from lemmas

Recall the function f_A from Definition VII.13:

$$(\forall x \in {}^{\omega}\omega)(\forall i \in \omega)\, f_A(x)(i) = \text{Rep}(C_{A,i})(x).$$

Theorem VII.28 (Borel Dominator Δ^1_2 Coding Theorem)**.** *For each $A \subseteq \omega$, whenever $g : {}^{\omega}\omega \to {}^{\omega}\omega$ is a Borel function satisfying*

$$(\forall x \in {}^{\omega}\omega)(\exists c \in \omega)\, f_A(x)(c) \leq g(x)(c),$$

then A is Δ^1_2 in any code for g.

Proof. Fix $A \subseteq \omega$, but assume without loss of generality that it is infinite and Δ^1_1 in every infinite subset of itself. Fix a Borel function $g : {}^{\omega}\omega \to {}^{\omega}\omega$ such that A is not Δ^1_2 in a fixed code for g. Also fix a well-founded tree $U \subseteq {}^{<\omega}\omega$ and for each $u \in U$ a Borel function $g_u : {}^{\omega}\omega \to {}^{\omega}\omega$ as was done at the beginning of this section. At this point, we may freely use the notation and lemmas used so far within this section. We will construct an $x \in {}^\omega\omega$ satisfying

$$(\forall i \in \omega)\, g(x)(i) < f(x)(i),$$

and the proof will be complete. Recall that $g = g_\emptyset$. As a result of Corollary VII.25,

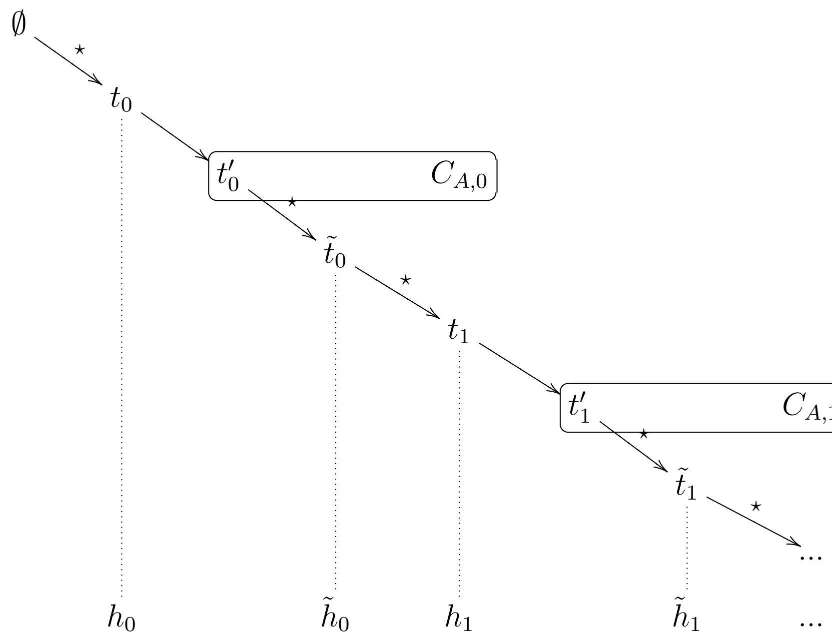
$$(\forall i, l \in \omega)[\Phi(\emptyset, l, i) \wedge \Xi^{\leq}(\emptyset, l, i) \wedge \Xi^{>}(\emptyset, l, i)].$$

We are also free to apply Lemma VII.26. We will construct a sequence of nodes

$$t_0 \sqsubseteq t_1 \sqsubseteq t_2 \sqsubseteq ...,$$

and our x will be $\bigcup_i t_i$.

The following diagram will guild the reader through this construction:



First, apply Lemma VII.26 and the fact that $(\forall l \in \omega)\, \Xi^{\leq}(\emptyset, l, 0)$ holds to get $t_0 \sqsupseteq^\star \emptyset$, $h_0 : {}^{<\omega}\omega \to \omega$, $l_0 \in \omega$, and η_0 such that η_0 is a winning strategy for Player II for the game

$$\mathcal{G}^{\leq}(t_0, h_0, \emptyset, l_0, 0).$$

At this point, we have ensured that $g(x)(0) \leq l_0$ (because we will apply the strategy η_0 infinitely many times during the construction of x, and all else that is done in the construction of the sequence x can be viewed as the moves of Player I in the game $\mathcal{G}^{\leq}(t_0, h_0, \emptyset, l_0, 0)$). Now, extend t_0 to a node $t_0' \sqsupseteq_{h_0} t_0$ such that $|t_0'| > l_0$ and $t_0' \in C_{A,0}$. This is possible because since $t_0 \sqsupseteq^\star \emptyset$, t_0 does not "hit" A. That is, $(\forall l < |t_0|)\, t_0(l) \notin A$. We have now decided that $f(x)(0) > l_0$. Next, apply the strategy η_0 to the pair (t_0', h_0) to get the pair $(\tilde{t}_0, \tilde{h}_0)$. Note that

$$(\tilde{t}_0, \tilde{h}_0) \leq^\star (t_0', h_0) \leq (t_0, h_0) \leq^\star (\emptyset, h_0).$$

Next, apply Lemma VII.26 and the fact that $(\forall l \in \omega)\, \Xi^{\leq}(\emptyset, l, 1)$ holds to get $(t_1, h_1) \leq^\star (\tilde{t}_0, \tilde{h}_0)$, $l_1 \in \omega$, and η_1 such that η_1 is a winning strategy for Player II for the game

$$\mathcal{G}^{\leq}(t_1, h_1, \emptyset, l_1, 1).$$

At this point, we have ensured that $g(x)(1) \leq l_1$ by the way we will construct the rest of x. Now, extend t_1 to a node $t_1' \sqsupseteq_{h_1} t_1$ such that $|t_1'| > l_1$ and $t_1' \in C_{A,1}$. This is possible because since $t_1 \sqsupseteq^\star t_0'$, t_1 does not hit A more than t_0' does. That is $\{l < |t_0'| : t_0'(l) \in A\}$ and $\{l < |t_1| : t_1(l) \in A\}$ both have size 1. We have now decided that $f(x)(1) > l_1$. Next, successively apply both η_0 and η_1 (the order does not matter) to the pair (t_1', h_1) to get the pair $(\tilde{t}_1, \tilde{h}_1)$. Note that

$$(\tilde{t}_1, \tilde{h}_1) \leq^\star (t_1', h_1) \leq (t_1, h_1) \leq^\star (\tilde{t}_0, \tilde{h}_0).$$

Continue this procedure forever. We have constructed a sequence of elements of $\mathbb{H}$

$$(t_0, h_0) \geq (t_0', h_0) \geq^\star (\tilde{t}_0, \tilde{h}_0) \geq^\star (t_1, h_1) \geq (t_1', h_1) \geq^\star (\tilde{t}_1, \tilde{h}_1) \geq^\star \ldots$$

such that

$$(\forall i \in \omega)\, t_i' \in C_{A,i},$$

a sequence of numbers

$$l_0, l_1, \ldots$$

such that

$$(\forall i \in \omega)\, |t'_i| > l_i,$$

and a sequence of strategies

$$\eta_0, \eta_1, \ldots$$

such that for each $i \in \omega$, η_i is a winning strategy for Player II for the game

$$\mathcal{G}^{\leq}(t_i, h_i, \emptyset, l_i, i).$$

Let

$$x := \bigcup_i t_i.$$

By the way the strategies η_i were applied, we have

$$(\forall i \in \omega)\, g(x)(i) \leq l_i.$$

At the same time since for each $i \in \omega$ we have $|t'_i| > l_i$, $t'_i \in C_{A,i}$, and $x \sqsupseteq t'_i$, we have

$$(\forall i \in \omega)\, l_i < f(x)(i).$$

Thus,

$$(\forall i \in \omega)\, g(x)(i) < f(x)(i),$$

and the proof is complete. □

7.7 Borel Challenge-Response Δ^1_2 Coding Theorem

The lemmas developed in the previous section allow us to prove a more general result. That is, we may replace the challenge-response relation $\langle {}^\omega\omega, {}^\omega\omega, \not\geq \rangle$ with any relation which satisfies the following property:

Definition VII.29. A challenge-response relation $\langle {}^\omega\omega, {}^\omega\omega, R \rangle$ has property $\mathcal{X}$ if there is a continuous function $c : {}^\omega\omega \to {}^\omega\omega$ satisfying

$$(\forall y \in {}^\omega\omega)\, \neg c(y) R y.$$

One can verify that essentially all of the challenge-response relations associated with cardinal characteristics of the continuum (are equivalent to ones which) have property $\mathcal{X}$. For example, fixing a standard bijection η from ${}^\omega\omega$ to $[\omega]^\omega$, we see that the relation $\langle {}^\omega\omega, {}^\omega\omega, S \rangle$ defined by

$$x_1 S x_2 \text{ iff } \eta(x_1) \text{ is split by } \eta(x_2)$$

has property $\mathcal{X}$. As another example, after fixing a standard way to code subtrees of ${}^{<\omega}\omega$ by elements of ${}^\omega\omega$, the relation $\langle {}^\omega\omega, {}^\omega\omega, W \rangle$ has property $\mathcal{X}$ where $x_1 W x_2$ iff either x_1 codes an ill-founded tree $T_2 \subseteq {}^{<\omega}\omega$, or x_1 and x_2 code well-founded trees $T_1 \subseteq {}^{<\omega}\omega$ and $T_2 \subseteq {}^{<\omega}\omega$ respectively and the rank of T_1 is less than or equal to the rank of T_2.

Out of all relations $R \subseteq {}^\omega\omega \times {}^\omega\omega$ which satisfy property $\mathcal{X}$, the weakest is non-equality of reals. Specifically, the reader can verify that R has property $\mathcal{X}$ iff there exists a morphism $\langle \phi_-, \phi_+ \rangle$ from $\langle {}^\omega\omega, {}^\omega\omega, R \rangle$ to $\langle {}^\omega\omega, {}^\omega\omega, \neq \rangle$ such that ϕ_- is continuous and ϕ_+ is the identity function. We will use this to state a remarkably strong corollary.

The proof of this next theorem is similar to that of Theorem VII.28, except we use $\mathcal{G}^=$ instead of $\mathcal{G}^\leq$ to get finer control over the behavior of $g(x)$. We will still use the sets $C_{A,i}$, but we will have to use a different function $f : {}^\omega\omega \to {}^\omega\omega$. Each node t'_i hits $C_{A,i}$ not at a level which is important, but such that the last value $t'_i(|t'_i|-1) \in A$ of t'_i is important.

Theorem VII.30 (Borel Challenge-Response Δ^1_2 Coding Theorem)**.** *Let $\langle {}^\omega\omega, {}^\omega\omega, R\rangle$ be a challenge-response relation and fix a continuous function $c : {}^\omega\omega \to {}^\omega\omega$ satisfying*

$$(\forall y \in {}^\omega\omega)\, \neg c(y) R y.$$

For each $A \subseteq \omega$, there is a Baire class one function $f : {}^\omega\omega \to {}^\omega\omega$ such that whenever $g : {}^\omega\omega \to {}^\omega\omega$ is a Borel function satisfying

$$(\forall x \in {}^\omega\omega)\, f(x) R g(x),$$

then A is Δ^1_2 in any code for g.

Proof. Fix $A \subseteq \omega$, but assume without loss of generality that it is Δ^1_1 in every infinite subset of itself. Fix a surjection $s : A \to {}^{<\omega}\omega$ such that for each $t \in {}^{<\omega}\omega$, $s^{-1}(t)$ is infinite. For each $i \in \omega$, let $C_{A,i} \subseteq {}^{<\omega}\omega$ be the cloud defined in Definition VII.13.

In the proof of Theorem VII.28, we defined $f(x)(i)$ to be the level where x "hits" $C_{A,i}$. Here, we will define $f(x)$ to be the concatenation of finite sequences, where the $(i+1)$-th finite sequence gets concatenated when x hits $C_{A,i}$, and that finite sequence is determined by the value of x at the level where x hits $C_{A,i}$. That is, we will define a function

$$\tilde{f} : {}^\omega\omega \to {}^\omega({}^{<\omega}\omega),$$

and then define $f : {}^\omega\omega \to {}^\omega\omega$ by

$$f(x) := \tilde{f}(x)(0)^\frown \tilde{f}(x)(1)^\frown ...$$

(and $f(x)$ is some arbitrary value if all but finitely many of the sequences $\tilde{f}(x)(0)$, $\tilde{f}(x)(1)$, ... are empty). Recall that given $x \in {}^\omega\omega$ and $i, l \in \omega$, $x \restriction l \in C_{A,i}$ implies

$x(l-1) \in A$. Define $\tilde{f}$ as follows:

$$\tilde{f}(x)(i) := \begin{cases} s(x(l-1)) & \text{if } x \restriction l \in C_{A,i}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Said another way, $\tilde{f}(x)(i)$ is the $s(x(l-1))$ such that $x(l-1)$ is the $(i+1)$-th element of A in the sequence

$$x = \langle x(0), x(1), ...\rangle$$

(and is $\emptyset$ if the sequence does not have at least $(i+1)$ elements of A).

Fix a Borel function $g : {}^\omega\omega \to {}^\omega\omega$ such that A is not Δ^1_2 in a fixed code for g. We will construct an $x \in {}^\omega\omega$ satisfying

$$\neg f(x) R g(x),$$

and the proof will be complete. At this point, we may freely use the notation and lemmas within the last section (because g is Borel, A is Δ^1_1 in any infinite subset of itself, and A is not Δ^1_2 in the code for g). Recall that $g = g_\emptyset$. As a result of Corollary VII.25,

$$(\forall i, l \in \omega)[\Phi(\emptyset, l, i) \wedge \Xi^{\leq}(\emptyset, l, i) \wedge \Xi^{>}(\emptyset, l, i)].$$

This allows us to apply Lemma VII.27, which is actually the only lemma we need. We will construct a sequence of nodes $t_0 \sqsubseteq t_1 \sqsubseteq ...$, and our x will be $\bigcup_i t_i$. The reader can use the same diagram which appears in the proof of Theorem VII.28 as a guide for this construction.

First, apply Lemma VII.27 to get $(t_0, h_0) \in \mathbb{H}$, $l_0 \in \omega$, and η_0 such that η_0 is a winning strategy for Player II for the game

$$\mathcal{G}^{=}(t_0, h_0, \emptyset, l_0, 0).$$

At this point, we have ensured that $g(x)(0) = l_0$ (because we will apply the strategy η_0 infinitely many times during the construction of x, and all else that is done in the construction of the sequence x can be viewed as the moves of Player I in the game $\mathcal{G}^{=}(t_0, h_0, \emptyset, l_0, 0)$).

We will now make use of the continuous function c. Let $v_0 \in {}^{<\omega}\omega$ be the longest finite sequence such that for each $y \in {}^{\omega}\omega$ extending $\langle l_0 \rangle$, $c(y)$ extends v_0. By hypothesis on the function s, $s^{-1}(v_0) \subseteq A$ is infinite. Also, $(\forall l < |t_0|)\, t_0(l) \notin A$, so we may extend t_0 to a node $t_0' \sqsupseteq_{h_0} t_0$ such that $t_0' \in C_{A,0}$ and s applied to the last element of the finite sequence t_0' is v_0. We have now decided that

$$\tilde{f}(x)(0) = v_0,$$

and hence $f(x)$ will extend v_0. Next, apply the strategy η_0 to the pair (t_0', h_0) to get the pair $(\tilde{t}_0, \tilde{h}_0)$. Note that

$$(\tilde{t}_0, \tilde{h}_0) \leq^{\star} (t_0', h_0) \leq (t_0, h_0) \leq^{\star} (\emptyset, h_0).$$

Next, apply Lemma VII.27 to get $(t_1, h_1) \leq^{\star} (\tilde{t}_0, \tilde{h}_0)$, $l_1 \in \omega$, and η_1 such that η_1 is a winning strategy for Player II for the game

$$\mathcal{G}^{=}(t_1, h_1, \emptyset, l_1, 1).$$

At this point, we have ensured that $g(x)(1) = l_1$ by the way we will construct the rest of x. We will again make use of the continuous function c. Let $v_1 \in {}^{<\omega}\omega$ be such that $v_0 ^\frown v_1$ is the longest finite sequence such that for all $y \in {}^{\omega}\omega$ extending $\langle l_0, l_1 \rangle$, $c(y)$ extends $v_0 ^\frown v_1$. By the hypothesis on the function s, $s^{-1}(v_1) \subseteq A$ is infinite. Also, since $t_1 \sqsupseteq^{\star} t_0'$, t_1 does not hit A more than t_0' does. Hence, $\{l < |t_1| : t_1(l) \in A\}$ has size 1. We can now easily extend t_1 to a node $t_1' \sqsupseteq_{h_1} t_1$ such that $t_1' \in C_{A,0}$ and s applied to the last element of the finite sequence t_1' is v_1. We have now decided that

$$\tilde{f}(x)(1) = v_1,$$

and hence $f(x)$ will extend $v_0{}^\frown v_1$. Next, successively apply both η_0 and η_1 (the order does not matter) to the pair (t_1, h_1) to get the pair $(\tilde{t}_1, \tilde{h}_1)$. Note that

$$(\tilde{t}_1, \tilde{h}_1) \leq^\star (t'_1, h_1) \leq (t_1, h_1) \leq^\star (\tilde{t}_0, \tilde{h}_0).$$

Continue this procedure forever. We have constructed a sequence of nodes

$$v_0, v_1, ... \in {}^{<\omega}\omega,$$

a sequence of elements of $\mathbb{H}$

$$(t_0, h_0) \geq (t'_0, h_0) \geq^\star (\tilde{t}_0, \tilde{h}_0) \geq^\star (t_1, h_1) \geq (t'_1, h_1) \geq^\star (\tilde{t}_1, \tilde{h}_1) \geq^\star ...$$

such that for each $i \in \omega$

$$t'_i \in C_{A,i}$$

and

$$s(t'_i(|t'_i| - 1)) = v_i,$$

a sequence of numbers

$$l_0, l_1, ...,$$

and a sequence of strategies

$$\eta_0, \eta_1, ...$$

such that for each $i \in \omega$, η_i is a winning strategy for Player II for the game

$$\mathcal{G}^=(t_i, h_i, \emptyset, l_i, i).$$

Let

$$x := \bigcup_i t_i.$$

By the way the strategies η_i were applied, we have

$$(\forall i \in \omega)\, g(x)(i) = l_i.$$

Define $y \in {}^\omega\omega$ to be

$$y := g(x) = \langle l_0, l_1, ...\rangle.$$

Now, y extends $\langle l_0\rangle$, so by the definition of v_0, $c(y)$ extends v_0. Similarly, since y extends $\langle l_0, l_1\rangle$, $c(y)$ extends $v_0^\frown v_1$. Continuing this argument we see that $c(y)$ extends $v_0^\frown v_1^\frown ...$. Since c is continuous, in fact $v_0^\frown v_1^\frown ...$ is an infinite sequence, so

$$c(y) = v_0^\frown v_1^\frown ....$$

At the same time, by the definition of $\tilde{f}$,

$$(\forall i \in \omega)\, \tilde{f}(x)(i) = v_i,$$

hence

$$f(x) = v_0^\frown v_1^\frown ....$$

Thus, we have shown

$$f(x) = c(g(x)).$$

By the hypothesis on c, we have

$$\neg f(x) R g(x).$$

This completes the proof. □

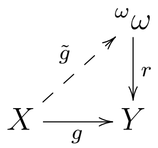
We now have a very strong corollary. The only work comes from considering arbitrary Polish spaces instead of ${}^\omega\omega$, which is generality we have suppressed up until this point.

Corollary VII.31. *Let X and Y be Polish spaces with X uncountable. For each $A \subseteq \omega$, there is a Borel $f : X \to Y$ such that whenever $g : X \to Y$ is Borel, then at least one of the following holds:*

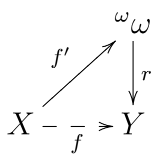
1) $(\exists x \in X)\, f(x) = g(x)$*;*

2) A *is* Δ^1_2 *in any code for* g*.*

Proof. Fix $A \subseteq \omega$. First, we claim that our choice of an arbitrary polish space Y as opposed to ${}^\omega\omega$ does not matter. That is, let $r : {}^\omega\omega \to Y$ be a continuous surjection. Given a Borel $g : X \to Y$, there is a Borel function $\tilde{g} : X \to {}^\omega\omega$ which makes the following diagram commute:



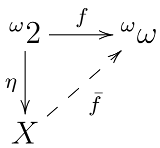
Furthermore, if A is Δ^1_2 in any code for $\tilde{g}$, then A is Δ^1_2 in any code for g. Suppose that we have proved that for some fixed Borel $f' : X \to {}^\omega\omega$, whenever $g' : X \to {}^\omega\omega$ is Borel and satisfies $(\forall x \in X)\, f'(x) \neq g'(x)$, then A is Δ^1_2 in any code for g'. Define f to make the following diagram commute:



Now suppose $g : X \to Y$ satisfies $(\forall x \in X)\, f(x) \neq g(x)$. We now have $(\forall x \in X)\, f'(x) \neq \tilde{g}(x)$. This implies A is Δ^1_2 in any code for $\tilde{g}$. This in turn implies that A is Δ^1_2 in any code for g. Thus, for the remainder of the proof, we may assume $Y = {}^\omega\omega$.

Next, we claim that the domain ${}^\omega\omega$ of the functions in the theorem above can be replaced with ${}^\omega 2$ at the cost of slightly increasing the complexity of f. The point is that every subset of ${}^{<\omega}\omega$ which is a cloud corresponds to a subset of ${}^{<\omega}2$ which is also a cloud. We leave this as an exercise to the reader, as the idea is simple but the details are messy.

The final piece of the puzzle is a standard fact: since X is an uncountable Polish space, there exists a Borel embedding $\eta : {}^{\omega}2 \to X$ such that whenever $f : {}^{\omega}2 \to {}^{\omega}\omega$ is Borel, there is a Borel function $\bar{f} : X \to {}^{\omega}\omega$ causing the following diagram to commute:



Furthermore, given Borel f and $\bar{f}$ causing this diagram to commute, if A is Δ^1_2 in any code for f, then A is Δ^1_2 in any code for $\bar{f}$.

We are now almost done. Let $f : {}^{\omega}2 \to {}^{\omega}\omega$ be Borel and such that whenever $g : {}^{\omega}2 \to {}^{\omega}\omega$ is Borel and satisfies $(\forall x \in {}^{\omega}2)\, f(x) \neq g(x)$, then A is Δ^1_2 in any code for g. Let $\bar{f}$ be the function given by the paragraph above (from f). Now suppose $\hat{g} : X \to {}^{\omega}\omega$ is Borel and satisfies $(\forall x \in X)\, \bar{f}(x) \neq \hat{g}(x)$. Let $g : {}^{\omega}2 \to {}^{\omega}\omega$ be the Borel function $\hat{g} \circ \eta$. We have $(\forall x \in {}^{\omega}2)\, f(x) \neq g(x)$, so A is Δ^1_2 in any code for g. By our comments at the end of the last paragraph, we have that A is Δ^1_2 in any code for $\hat{g}$. $\square$

Note that in the theorem, instead of considering the set $\mathcal{F}$ of functions $\tilde{f}$ whose corresponding f is Borel, we could have considered the set $\mathcal{F}'$ of *Borel functions from* ${}^{\omega}\omega$ *to Ord* where Ord is given the discrete topology. Our proof of the theorem pushes through to give us functions $\phi_- : \mathcal{P}(\omega) \to \mathcal{F}'$ and $\phi_+ : \mathcal{F}' \to \mathcal{P}(\omega)$ with the same properties as above. This ordering is closer to what is studied in [11].

One might further hope that there is an application to the *Steel Hierarchy of Norms* (also called the FPT Hierarchy for "First Periodicity Theorem") [35]. That is, giving the ordinals the discrete topology, one might hope to show that for each $A \subseteq \omega$ and each countable limit ordinal α that is the image of a Borel function, there exists a Borel $\varphi : {}^{\omega}\omega \to \alpha$ such that if $\psi : {}^{\omega}\omega \to \alpha$ is Borel and there exists a continuous $i : {}^\omega\omega \to {}^\omega\omega$ satisfying

$$(\forall x \in {}^\omega\omega)\, \varphi(x) \leq \psi(i(x)),$$

then A is constructible from a "Borel code" for ψ. Currently, our arguments only show we can ensure that A is constructible from the pair consisting of a "Borel code" for ψ and a Borel code for i. Moreover, it can be checked that our particular encoding scheme cannot accomplish this stronger goal. The existential quantification of the continuous function i seems to drastically change the situation.

CHAPTER VIII

Conclusion

Let us end by asking some questions.

8.1 Some Questions

We have seen various encoding schemes for functions from a set X to κ where κ is an infinite cardinal and $|X| \geq 2^\kappa$. We ask the general question of whether similar encodings can exist but assuming $|X| < 2^\kappa$. For example, assuming $\neg$CH, what is the collection $\mathcal{C} \subseteq \mathcal{P}(\omega)$ of sets $A \subseteq \omega$ for which there exists an $f : \omega_1 \to \omega$ such that if $g : \omega_1 \to \omega$ satisfies $f \leq g$, then $A \in L[g]$? By Section 2.8, $\mathcal{C}$ contains all Δ^1_1 subsets of ω (because those sets can be encoded into functions from ω to ω, let alone functions from ω_1 to ω). Can $\mathcal{C}$ ever be strictly larger than Δ^1_1? Is it always strictly larger? The following is related, because Sacks forcing (to add a single real) is in some sense the gentlest way to add a real. Note that by Theorem V.35, a model which affirmatively answers the following question must satisfy $\neg$CH.

Question VIII.1. *Is it consistent that Sacks forcing is weakly (ω_1, ω)-distributive?*

Taking a step back, we ask what morphisms exist from combinatorial challenge-response relations to various recursion-theoretic orderings on $\mathcal{P}(\omega)$ and larger structures such as $\mathcal{P}(\mathbb{R})$. The purpose of such questions is to lower bound the inherent complexity within challenge-response relations that arise in practice (such as the poset used in the definition of Borel boundedness, which is what we did).

Here is the most interesting question: can Theorem VII.30 be generalized beyond Borel functions? The following definition seems appropriate. We use $\le^*$ as the relation because it is concrete but simultaneously captures the main idea for all reasonable relations (our evidence being that the proof of Theorem VII.30 is only a slight generalization of the proof of Theorem VII.28). Let us say that a pointclass Γ of functions from ${}^\omega\omega$ to ${}^\omega\omega$ has the *encoding property* if for each $A \subseteq \omega$, there exists a Borel function $f : {}^\omega\omega \to {}^\omega\omega$ such that whenever $g : {}^\omega\omega \to {}^\omega\omega$ is in Γ and

$$(\forall x \in {}^\omega\omega)\, f(x) \le^* g(x),$$

then A is in some countable set associated to g. By "some countable set associated to g", we have in mind "$A \in \text{HOD}(c)$ where c is any code for g" (assuming both AD and that there is a canonical way to code elements of Γ by reals). We require f to be Borel simply because we believe that using more complicated functions to encode reals is unnecessary. Indeed, we believe the encoding $A \mapsto f_A$ given by Definition VII.13 suffices. When we made the generalization from Baire class one dominators to Borel dominators, the same encoding sufficed. We naturally expect this pattern to continue.

The problem becomes to prove from additional set theoretic axioms (determinacy or large cardinals) that larger and larger pointclasses have the encoding property. Just as Lebesgue measurability and the property of Baire are regularity properties, so too should be the encoding property. What is the relationship between the encoding property and other regularity properties? Since essentially all known regularity properties follow from determinacy, we should expect the same for the encoding property.

It would be interesting if the encoding property coexists with determinacy, without there being a short proof that $\mathbf{\Delta}^1_1$ has the encoding property from Borel determinacy. It is possible that even with large cardinals, $\mathbf{\Delta}^1_1$ is the largest class which can be proven to have the encoding property. This would explain the apparent difficulty in reworking the proof that $\mathbf{\Delta}^1_1$ has the encoding property to use Borel determinacy. We suspect that the encoding property has more in common with the Ramsey property than with the perfect set property, the Lebesgue measurability property, or the Baire property.

Finally, let us take a leap out of the area of this thesis and conjecture that the axiom of determinacy implies many more encoding theorems exist. If we have functions $f, g : {}^\omega\omega \to {}^\omega\omega$ and a relation $R \subseteq {}^\omega\omega \times {}^\omega\omega$ that is a prewellordering of ${}^\omega\omega$ of order type α such that

$$(\forall x \in {}^\omega\omega)\, f(x) R g(x),$$

then this is *similar* to having functions $\tilde{f}, \tilde{g} : {}^\omega\omega \to \alpha$ satisfying

$$(\forall x \in {}^\omega\omega)\, \tilde{f}(x) \leq \tilde{g}(x).$$

Question VIII.2. *Assume* AD*. For each limit ordinal $\alpha < \Theta$ and for each $A \subseteq {}^\omega\omega$, is there is a function $f : {}^\omega\omega \to \alpha$ such that whenever $g : {}^\omega\omega \to \alpha$ satisfies $f \leq g$, then $A \in L({}^\omega\omega, g)$?*

This is a question about subsets of ${}^\omega\omega$ rather than subsets of ω, but we cross our fingers and conjecture that it is true.

APPENDICES

APPENDIX A

Absoluteness of Domination for Nice Functions

The following observations are natural for investigating the domination ordering of Borel functions.

Definition A.1. Given a transitive model M of ZF+DC and a Borel code $c \in ({}^{\omega}\omega)^M$, let c_M refer to the object in M coded by c. We use Borel codes interchangeably for subsets of a Polish space or for functions from one Polish space to another.

Given a real c, it is a $\mathbf{\Pi}^1_1$ property whether or not c is a Borel code [26]. That is, the set $\mathrm{BC} \subseteq {}^{\omega}\omega$ of Borel codes is $\mathbf{\Pi}^1_1$. The following illustrates the absoluteness of membership in a Borel set:

Fact A.2. *Let X be a Polish space. There is a $\mathbf{\Sigma}^1_1$ set $P \subseteq X \times {}^{\omega}\omega$ and a $\mathbf{\Pi}^1_1$ set $Q \subseteq X \times {}^{\omega}\omega$ such that if $c \in {}^{\omega}\omega$ is a Borel code, then*

$$x \in c_V \Leftrightarrow (x, c) \in P \Leftrightarrow (x, c) \in Q$$

for all $x \in X$.

For the remainder of this section, let M be a transitive model of ZF + DC. Let X and Y be Polish spaces. Combining the fact above with $\mathbf{\Pi}^1_1$ absoluteness, we immediately have the following:

Corollary A.3. *Let a, b, c be Borel codes in M. The following hold:*

1) $a_M \subseteq b_M$ *iff* $a_V \subseteq b_V$*;*

2) $a_M = b_M$ *iff* $a_V = b_V$*;*

3) $a_M = b_M \cup c_M$ *iff* $a_V = b_V \cup c_V$*;*

4) $a_M = b_M \cap c_M$ *iff* $a_V = b_V \cap c_V$*;*

5) $a_M = \emptyset$ *iff* $a_V = \emptyset$*.*

Another useful consequence of $\mathbf{\Pi}^1_1$ absoluteness is this:

Corollary A.4. *If* c *is a Borel code in* M *for a subset of* X*, then* $c_M = c_V \cap M$*.*

A consequence of $\mathbf{\Pi}^1_2$ absoluteness is this:

Corollary A.5. *Suppose* $\omega_1 \subseteq M$ *(so* $\mathbf{\Pi}^1_2$ *formulas are absolute between* M *and* V*). If* c *is a Borel code in* M *for a subset of* $X \times Y$*, then* $(c_M$ *is a function*$)^M$ *iff* c_V *is a function. Furthermore, if* $(c_M$ *is a function*$)^M$*, then* $c_M = c_V \restriction M$*.*

The following is relevant to our investigation:

Proposition A.6. *If* a *and* b *are Borel codes in* M *for functions from* X *to* ω*, then* $(a_M \leq b_M)^M$ *iff* $a_V \leq b_V$*.*

Proof. Fix such a and b. By $\mathbf{\Pi}^1_1$ absoluteness,

$$\begin{aligned} &M \models (\forall x \in \mathcal{N})(\forall n, m \in \omega)[(x,n) \in a_M \wedge (x,m) \in b_M \rightarrow n \leq m] \\ \text{iff} \quad &V \models (\forall x \in \mathcal{N})(\forall n, m \in \omega)[(x,n) \in a_V \wedge (x,m) \in b_V \rightarrow n \leq m], \end{aligned}$$

which is what we want. □

For eventual domination, we have an analogous result:

Proposition A.7. *If* a *and* b *are Borel codes in* M *for functions from* ${}^\omega\omega$ *to* ${}^\omega\omega$*, then* $(a_M \leq^* b_M)^M$ *iff* $a_V \leq^* b_V$*.*

APPENDIX B

Tameness of Cardinal Characteristics

Zapletal has defined a notion of a cardinal characteristic being *tame*. Tame characteristics have some desirable properties, and both $\operatorname{cf}\mathcal{B}_\alpha(\omega, \leq)$ and $\operatorname{cf}\mathcal{B}_\alpha({}^\omega\omega, \leq^*)$ for $\alpha \leq \omega_1$ are tame. The following is from [46]:

Definition B.1. A cardinal characteristic is *tame* if it is defined as

$$\min\{|A| : A \subseteq {}^\omega\omega \wedge \phi(A) \wedge (\forall x \in {}^\omega\omega)(\exists y \in A)\, xRy\}$$

where $R \subseteq {}^\omega\omega \times {}^\omega\omega$ is projective and the quantifiers of $\phi(A)$ are restricted to the set A or to the set of natural numbers.

For our purposes, this is the crucial property of tame characteristics:

Theorem B.2. *Suppose that there is a proper class of measurable Woodin cardinals. If $\mathfrak{x}$ is a tame cardinal invariant such that $\mathfrak{x} < 2^\omega$ holds in some set forcing extension, then $\mathfrak{x} < 2^\omega$ holds in the iterated Sacks extension.*

Proof. See [46]. □

Thus, when we investigate a tame cardinal characteristic which we do not yet know is provably (in ZFC) equal to 2^ω, analyzing the effect of iterated Sacks forcing is extremely useful. Indeed, we can learn much by adding a *single* Sacks real.

We will explain why $\operatorname{cf}\mathcal{B}_{\omega_1}({}^{\omega}\omega, \leq^*)$ is tame (a similar reason applies to both $\operatorname{cf}\mathcal{B}_{\alpha}(\omega, \leq)$ and $\operatorname{cf}\mathcal{B}_{\alpha}({}^{\omega}\omega, \leq^*)$ for each $\alpha \leq \omega_1$). Let $\mathrm{BC} \subseteq {}^{\omega}\omega$ be the set of codes for Borel functions from ${}^{\omega}\omega$ to ${}^{\omega}\omega$. Certainly, BC is projective. Let $R \subseteq {}^{\omega}\omega \times {}^{\omega}\omega$ be such that xRy iff either $x \notin \mathrm{BC}$, or simultaneously $x \in \mathrm{BC}$, $y \in \mathrm{BC}$, and the function coded by y pointwise eventually dominates the function coded by x. The relation R is projective. Finally, letting $\phi(A)$ be identically true, we have

$$\operatorname{cf}\mathcal{B}_{\omega_1}({}^{\omega}\omega, \leq^*) = \min\{|A| : A \subseteq {}^{\omega}\omega \wedge \phi(A) \wedge (\forall x \in {}^{\omega}\omega)(\exists y \in A)\, xRy\},$$

so $\operatorname{cf}\mathcal{B}_{\omega_1}({}^{\omega}\omega, \leq^*)$ is tame.

APPENDIX C

Sacks Forcing and Fusion

Within this short section, we will define Sacks forcing and provide a lemma that will help to perform *fusion*. We use this in Section 6.2.

Definition C.1. A tree $p \subseteq {}^{<\omega}2$ is *perfect* if it is nonempty and for each $t \in p$, there are incomparable $t_1, t_2 \in p$ extending t. Sacks forcing $\mathbb{S}$ is the poset of all perfect trees $p \subseteq {}^{<\omega}2$ where $p_1 \leq p_2$ iff $p_1 \subseteq p_2$.

Given $p_1, p_2 \in \mathbb{S}$, $p_1 \perp p_2$ means that p_1 and p_2 are incompatible.

Definition C.2. Let $p \subseteq {}^{<\omega}2$ be a perfect tree. A node $t \in p$ is called a *branching node* if $t^\frown 0, t^\frown 1 \in p$. Stem$(p)$ is the unique branching node t of p such that all elements of p are comparable to t. A node $t \in p$ is said to be an *n-th branching node* if it is a branching node and there are exactly n branching nodes that are proper initial segments of it. In particular, Stem(p) is the unique 0-th branching node of p. Given Sacks conditions p, q, we write $q \leq_n p$ if $q \leq p$ and all of the k-th branching nodes, for $k \leq n$, of p are in q and are branching nodes.

Lemma C.3 (Fusion Lemma)**.** *Let $\langle p_n : n \in \omega \rangle$ be a sequence of Sacks conditions such that*

$$p_0 \geq_0 p_1 \geq_1 p_2 \geq_2 \ldots.$$

Then $p_\omega := \bigcap_{n\in\omega} p_n$ is a Sacks condition below each p_n.

Proof. This is standard and can be found in introductory presentations of Sacks forcing. See, for example, [24]. □

The sequence $\langle p_n : n \in \omega \rangle$ in the above lemma is known as a *fusion sequence*. The following will help in the construction of fusion sequences.

Lemma C.4 (Fusion Helper Lemma)**.** *Let $\mathbb{S}$ be Sacks forcing. Let $R : {}^{<\omega}2 \to \mathbb{S}$ be a function with the following properties:*

1) $(\forall s_1, s_2 \in {}^{<\omega}2)$ $s_2 \sqsupseteq s_1$ implies $R(s_2) \leq R(s_1)$;

2) $(\forall s \in {}^{<\omega}2)$ $Stem(R(s^\frown 0)) \perp Stem(R(s^\frown 1))$.

For each $n \in \omega$, let p_n be the Sacks condition

$$p_n := \bigcup \{R(s) : s \in {}^{n}2\}.$$

Then

$$R(\emptyset) = p_0 \geq p_1 \geq_0 p_2 \geq_1 p_3 \geq_2 \dots$$

is a fusion sequence.

Proof. Consider any $n \geq 1$. Certainly $p_n \supseteq p_{n+1}$, because for each $s \in {}^{n}2$, $R(s) \supseteq R(s^\frown 0) \cup R(s^\frown 1)$. To show that $p_n \geq_{n-1} p_{n+1}$, consider a k-th branching node t of p_n for some $k \leq n-1$. One can check that there is some $s \in {}^{k}2$ such that t is the largest common initial segment of $\mathrm{Stem}(R(s^\frown 0))$ and $\mathrm{Stem}(R(s^\frown 1))$. Since

$$\mathrm{Stem}(R(s^\frown 0)) \cup \mathrm{Stem}(R(s^\frown 1)) \subseteq R(s^\frown 0) \cup R(s^\frown 1) \subseteq p_{n+1},$$

we have that t is a branching node of p_{n+1}. Thus, we have shown that for each $k \leq n-1$, each k-th branching node of p_n is a branching node of p_{n+1}. Hence, $p_n \geq_{n-1} p_{n+1}$. □

In the proposition above, if we define

$$q := \bigcap_n p_n,$$

then we have the representation

$$q = \{t \in {}^{<\omega}2 : t \sqsubseteq \mathrm{Stem}(R(s)) \text{ for some } s \in {}^{<\omega}2)\},$$

and every $x \in [q]$ is uniquely determined by the set of $s \in {}^{<\omega}2$ for which $\mathrm{Stem}(R(s)) \sqsubseteq x$.

APPENDIX D

Sacks Forcing and Continuous Reading of Names

This section may be useful to anyone who works with Sacks forcing (especially the final proposition). The following is commonly called "continuous readings of names":

Proposition D.1. *Let p be a Sacks condition. Let $\dot{\tau}$ be such that $p \Vdash (\dot{\tau} \in {}^{\omega}\omega)$. Then there is some $q \leq p$ and a name $\dot{g}$ for a continuous function from $[q]$ to ${}^{\omega}\omega$, which is coded by a Borel code in V, satisfying*

$$q \Vdash (\dot{g}(\dot{\sigma}) = \dot{\tau})$$

where $\dot{\sigma}$ is the canonical name for the generic real.

Proof. We will define a function $R : {}^{<\omega}2 \to \mathbb{S}$ satisfying conditions 1 and 2 of Lemma C.4. At the same time, we will also define a function $N : {}^{<\omega}2 \to {}^{<\omega}\omega$. We will define these by induction on the length of their input. Let $R(\emptyset) = p$ and $N(\emptyset) = \emptyset$. Now, suppose that $s \in {}^{n}2$ and we have defined $R(s)$ and $N(s)$. Let $R(s^\frown 0)$, $R(s^\frown 1)$, $N(s^\frown 0)$, and $N(s^\frown 1)$ be defined in any way such that the following are satisfied:

1) $R(s^\frown 0), R(s^\frown 1) \leq R(s)$;

2) $\mathrm{Stem}(R(s^\frown 0)) \perp \mathrm{Stem}(R(s^\frown 1))$;

3) $|N(s^\frown 0)|, |N(s^\frown 1)| \geq n + 1$;

4) $R(s^\frown i) \Vdash \overbrace{N(s^\frown i)}^{\vee} \sqsubseteq \dot{\tau}$ for $i = 0, 1$.

It is clear that such values exist. That is, we may initially pick $R(s^\frown 0)$ and $R(s^\frown 1)$ to be strengthenings of $R(s)$ with incompatible stems, and then strengthen them more to decide the first $n+1$ values of $\dot{\tau}$. This completes the definition of R and N.

By 1 and 2 above, the function R satisfies the conditions of Lemma C.4. Let q be the intersection of the fusion sequence given by that lemma. Let g be the continuous function in V satisfying

$$(\forall x \in [q])(\forall s \in {}^{<\omega}2)[\mathrm{Stem}(R(s)) \sqsubseteq x \rightarrow N(s) \sqsubseteq g(x)].$$

Let $\dot{g}$ be a name for the unique continuous function in the forcing extension which extends g. Note that the continuous function in the forcing extension is coded by a Borel code in V (which is in fact the Borel code for g in V). We have

$$1 \Vdash (\forall x \in [\check{q}])(\forall s \in {}^{<\omega}2)[\mathrm{Stem}(\check{R}(s)) \sqsubseteq x \rightarrow \check{N}(s) \sqsubseteq \dot{g}(x)].$$

Since $q \Vdash \dot{\sigma} \in [\check{q}]$, we have

$$q \Vdash (\forall s \in {}^{<\omega}2)[\mathrm{Stem}(\check{R}(s)) \sqsubseteq \dot{\sigma} \rightarrow \check{N}(s) \sqsubseteq \dot{g}(\dot{\sigma})].$$

Consider any $n \in \omega$ and $s \in {}^{n}2$. By the definition of $\dot{\sigma}$,

$$R(s) \Vdash \mathrm{Stem}(\check{R}(\check{s})) \sqsubseteq \dot{\sigma}.$$

This means

$$q \cap R(s) \Vdash \check{N}(\check{s}) \sqsubseteq \dot{g}(\dot{\sigma}).$$

On the other hand, $|N(s)| \geq n$ and $R(s) \Vdash \check{N}(\check{s}) \sqsubseteq \dot{\tau}$, so

$$q \cap R(s) \Vdash \dot{g}(\dot{\sigma}) \restriction \check{n} = \dot{\tau} \restriction \check{n}.$$

Let $p_n := \bigcup\{R(s) : s \in {}^n2\}$. Since any extension of $q \cap p_n$ can be strengthened to an extension of $q \cap R(s)$ for some $s \in {}^n2$, by density we have

$$q \cap p_n \Vdash \dot{g}(\dot{\sigma}) \restriction \check{n} = \dot{\tau} \restriction \check{n}.$$

Since $q \leq q \cap p_n$ for all n,

$$q \Vdash (\forall n \in \omega)\, \dot{g}(\dot{\sigma}) \restriction n = \dot{\tau} \restriction n.$$

Hence,

$$q \Vdash \dot{g}(\dot{\tau}) = \dot{\tau},$$

and we are done. □

Something special about Sacks forcing is that we can get the function $\dot{g}$ to be one-to-one as long as $p \Vdash (\dot{\sigma} \notin \check{V})$:

Proposition D.2. *Let p be a Sacks condition. Let $\dot{\tau}$ be such that $p \Vdash (\dot{\tau} \in {}^\omega\omega)$ and $p \Vdash (\dot{\tau} \notin \check{V})$. Then there is some $q \leq p$ and a name $\dot{g}$ for a continuous and one-to-one function from $[q]$ to ${}^\omega\omega$, where the function is coded by a Borel code in V, satisfying*

$$q \Vdash (\dot{g}(\dot{\sigma}) = \dot{\tau})$$

where $\dot{\sigma}$ is the canonical name for the generic real.

Proof. We may perform the same construction in the above proof but also with the requirement that

$$(\forall s \in {}^{<\omega}2)\, N(s^\frown 0) \perp N(s^\frown 1).$$

We will show that the resulting function $\dot{g}$ is injective. Suppose $\dot{a}$ and $\dot{b}$ are names satisfying $1 \Vdash \dot{a} \in [\check{q}]$, $1 \Vdash \dot{b} \in [\check{q}]$, and $1 \Vdash \dot{a} \neq \dot{b}$. We will show that

$$\{r \in \mathbb{S} : r \Vdash \dot{g}(\dot{a}) \neq \dot{g}(\dot{b})\}$$

is dense in $\mathbb{S}$, which will establish that $1 \Vdash \dot{g}(\dot{a}) \neq \dot{g}(\dot{b})$.

Pick any $r \in \mathbb{S}$. There exists some $r' \leq r$ and $s \in {}^{<\omega}2$ satisfying $r' \Vdash R(s^\frown 0) \sqsubseteq \dot{a}$ and $r' \Vdash R(s^\frown 1) \sqsubseteq \dot{b}$. Using the definition of $\dot{g}$ gives us $r' \Vdash N(s^\frown 0) \sqsubseteq \dot{g}(\dot{a})$ and $r' \Vdash N(s^\frown 1) \sqsubseteq \dot{g}(\dot{b})$. Since $N(s^\frown 0) \perp N(s^\frown 1)$, we have $r' \Vdash \dot{g}(\dot{a}) \neq \dot{g}(\dot{b})$. This completes the proof. □

We can generalize this proposition to handle countably many reals simultaneously. This requires us to enhance the argument and there is no clear way to deduce it from the proposition above (such as using a scheme to code countably many reals into a single real)

Proposition D.3. *Let p be a Sacks condition. Let $\dot{\tau}$ be a name satisfying $p \Vdash (\dot{\tau} : \omega \times \omega \to \omega)$. For each $n \in \omega$, let $\dot{\tau}_n$ be a name for the function $i \mapsto \dot{\tau}(n, i)$ in the extension. Suppose that for each $n \in \omega$, $p \Vdash (\dot{\tau}_n \notin \check{V})$. Then there is some $q \leq p$ and a name $\dot{g}$ for a function from $\omega \times [q]$ to ${}^\omega\omega$, which is coded by a Borel code in V, satisfying*

$$q \Vdash (\forall n \in \omega)[(x \mapsto \dot{g}(n, x)) \textit{ is continuous and one-to-one}]$$

and

$$q \Vdash (\forall n \in \omega)\, \dot{g}(n, \dot{\sigma}) = \dot{\tau}_n$$

where σ is the canonical name for the generic real.

Proof. We will define a function $R : {}^{<\omega}2 \to \mathbb{S}$ satisfying conditions 1 and 2 of Lemma C.4. Using the proposition above with condition p and name $\dot{\tau}_0$, let $R(\emptyset)$ be p and let $\dot{g}_0$ be the name for the function given by that proposition. That is, $\dot{g}_0$ is a name for a continuous and one-to-one function from $[R(\emptyset)]$ to ${}^\omega\omega$ for which

$$R(\emptyset) \Vdash \dot{g}_0(\dot{\sigma}) = \dot{\tau}_0.$$

Next, let r_0 and r_1 be two extensions of $R(\emptyset)$ with incompatible stems. We may apply the proposition above to strengthen r_0 to some condition r_0' and get a name $\dot{h}_0$ for a continuous and one-to-one function from $[r_0']$ to ${}^\omega\omega$ for which $r_0' \Vdash \dot{h}_0(\dot{\sigma}) = \dot{\tau}_1$. Similarly, we may strengthen r_1 to some condition r_1' and get a name $\dot{h}_1$ for a continuous and one-to-one function from $[r_1']$ to ${}^\omega\omega$ for which $r_1' \Vdash \dot{h}_1(\dot{\sigma}) = \dot{\tau}_1$. For ease of notation, let h_0 and h_1 be the functions $\dot{h}_0$ and $\dot{h}_1$ respectively restricted to V. Now, since $[r_0']$ and $[r_1']$ are disjoint closed sets and h_0 and h_1 are continuous, the function $h : [r_0'] \cup [r_1'] \to {}^\omega\omega$ defined by

$$h(x) := \begin{cases} h_0(x) & \text{if } x \in [r_0'], \\ h_1(x) & \text{if } x \in [r_1'] \end{cases}$$

is continuous. However, h need not be one-to-one. Here is how we can fix this: pick any $y_0 \in \mathrm{Im}(h_0)$ and $y_1 \in \mathrm{Im}(h_1)$ such that $y_0 \neq y_1$ (y_0 can be picked arbitrarily, and a y_1 must exist because $[r_1]$ has more than one element and h_1 is one-to-one). Let $U_0 \ni y_0$ and $U_1 \ni y_1$ be disjoint open subsets of ${}^\omega\omega$. Since h_0 is continuous, we may strengthen r_0' to some r_0'' so that $h_0``([r_0'']) \subseteq U_0$. Similarly, we many strengthen $r_1'$ to some $r_1''$ so that $h_1``([r_1'']) \subseteq U_1$. Define $R(\langle 0\rangle) := r_0''$ and $R(\langle 1\rangle) := r_1''$. Let $g_1 : [R(\langle 0\rangle)] \cup [R(\langle 1\rangle)] \to {}^\omega\omega$ be the continuous function $h \restriction [R(\langle 0\rangle)] \cup [R(\langle 1\rangle)]$. By construction, g_1 is continuous and one-to-one. If $\dot{g}_1$ is the name for the continuous function with the same Borel code, then

$$R(\langle 0\rangle) \Vdash \dot{g}_1(\dot{\sigma}) = \dot{\tau}_1$$

and

$$R(\langle 1\rangle) \Vdash \dot{g}_1(\dot{\sigma}) = \dot{\tau}_1,$$

so

$$R(\langle 0\rangle) \cup R(\langle 1\rangle) \Vdash \dot{g}_1(\dot{\sigma}) = \dot{\tau}_1.$$

We may continue like this to define $R : {}^{<\omega}2 \to \mathbb{S}$ along with, for each $n \in \omega$, a name $\dot{g}_n$ for a continuous and one-to-one function from $\bigcup\{R(s) : s \in {}^n2\}$ to ${}^\omega\omega$ so that

$$\bigcup\{R(s) : s \in {}^n2\} \Vdash \dot{g}_n(\dot{\sigma}) = \dot{\tau}_n.$$

We may now take the intersection of the fusion sequence:

$$q := \bigcap_n \bigcup\{R(s) : s \in {}^n2\}.$$

For each $n \in \omega$, we have

$$q \Vdash \dot{g}_n(\dot{\sigma}) = \dot{\tau}_n.$$

Let $\dot{g}$ be the canonical name for the function from $\omega \times [q]$ to ${}^\omega\omega$ so that

$$1 \Vdash (\forall n \in \omega)\dot{g}(n, x) = \dot{g}_n(x).$$

For each $n \in \omega$,

$$1 \Vdash \text{the function } x \mapsto \dot{g}(n, x) \text{ is continuous and one-to-one}$$

because

$$1 \Vdash [q] \subseteq \mathrm{Dom}(\dot{g}_n)$$

and

$$1 \Vdash \dot{g}_n \text{ is continuous and one-to-one.}$$

Hence,

$$q \Vdash (\forall n \in \omega) \text{ the function } x \mapsto \dot{g}(n, x) \text{ is continuous and one-to-one.}$$

Furthermore, it can be checked that there is a Borel code in V that codes the function $\dot{g}$ in the extension. This completes the proof. □

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] A. Blass. Needed reals and recursion in generic reals. *Ann. Pure Appl. Logic*, 109:77–88, 2001.

[2] A. Blass. Combinatorial cardinal characteristics of the continuum. In Matthew Foreman and Akihiro Kanamori, editors, *Handbook of Set Theory*, pages 395–489. Springer, New York, NY, 2010.

[3] C. Boykin and S. Jackson. Borel boundedness and the lattice rounding property. In *Advances in Logic*, volume 425, pages 113–126. Amer. Math. Soc., Providence, RI, 2007.

[4] M. Burke and M. Magidor. Shelah's pcf theory and its applications. *Ann. Pure Appl. Logic*, 50:207–254, 1990.

[5] W. Comfort and S. Negrepontis. *The Theory of Ultrafilters*. Springer-Verlag, New York - Heidelberg, 1974.

[6] J. Cummings. Compactness and incompactness phenomena in set theory. In *Logic Colloquium 2001*, volume 20, pages 139–150. Assoc. Symbol. Logic, Urbana, IL, 2005.

[7] J. Cummings and S. Shelah. Cardinal invariants above the continuum. *Ann. Pure Appl. Logic*, 75:251–268, 1995.

[8] M. Daguenet. Emploi des filtres sur N dans l'étude descriptive des fonctions. *Fund. Math.*, 95:11–33, 1977.

[9] N. Dobrinen. *Generalized Weak Distributive Laws in Boolean Algebras and Issues Related to a Problem of non Neumann Regarding Measurable Algebras*. PhD thesis, University of Minnesota, 2001.

[10] P. Dordal. Towers in $[\omega]^\omega$ and ${}^\omega\omega$. *Ann. Pure Appl. Logic*, 45:247–276, 1989.

[11] J. Duparc. The Steel hierarchy of ordinal valued Borel mappings. *J. Symbolic Logic*, 68:187–234, 2003.

[12] M. Elekes and K. Kunen. Transfinite sequences of continuous and Baire class 1 functions. *Proc. Amer. Math. Soc.*, 131:2453–2457, 2003.

[13] M. Elekes and Z. Vidnyánszky. Characterization of order types of pointwise linearly ordered families of Baire class 1 functions. 2014.

[14] R. Engelking and M. Karlowicz. Some theorems of set theory and their topological consequences. *Fund. Math.*, 57:275–285, 1965.

[15] I. Farah. OCA and towers in P(N)/fin. *Comment. Math. Univ. Carolin.*, 37:861–866, 1996.

[16] G. Fichtenholz and L. Kantorovitch. Sur les opérations lineaires dans l'espace des fonctions bornées. *Studia Math.*, 5:69–98, 1934.

[17] D. Fremlin. Real-valued measurable cardinals. In Haim Judah, editor, *Israel Mathematical Conference Proceedings: Set Theory of the Reals*, volume 6, pages 151–304. Americal Mathematical Society, 1993.

[18] M. Gitik. Prikry-type forcings. In Matthew Foreman and Akihiro Kanamori, editors, *Handbook of Set Theory*, pages 1351–1447. Springer, New York, NY, 2010.

[19] L. Harrington. Long projective wellorderings. *Ann. Math. Logic*, 12:1–24, 1977.

[20] F. Hausdorff. Über zwei Sätze von G. Fichtenholz und L. Kantorovitch. *Studia Math.*, 6:18–19, 1936.

[21] S. Hechler. On the existence of certain cofinal subsets of ${}^{\omega}\omega$. *Proc. Symp. Pure Math.*, 13:155–173, 1974.

[22] P. Hinman. *Recursion-Theoretic Hierarchies.* Springer-Verlag, New York, NY, 1978.

[23] M. Holz, K. Steffens, and E. Weitz. *Introduction to Cardinal Arithmetic.* Birkhäuser-Verlag, Basel, 1999.

[24] T. Jech. *Multiple forcing.* Cambridge University Press, Cambridge, 1986.

[25] T. Jech. Distributive laws. In R. Bonnet and J.D. Monk, editors, *Handbook of Boolean Algebra.* North-Holland Publishing Co., Amsterdam, 1989.

[26] T. Jech. *Set Theory, the Third Millennium Edition.* Springer-Verlag, Berlin, 2002.

[27] T. Jech and K. Prikry. Cofinality of the partial ordering of functions from ω_1 to ω under eventual domination. *Math. Proc. Cambridge Philos. Soc.*, 95:25–32, 1984.

[28] C. Jockusch. Uniformly introreducible sets. *J. Symbolic Logic*, 33:521–536, 1968.

[29] A. Kamburelis. On the weak distributivity game. *Ann. Pure Appl. Logic*, 66:19–26, 1994.

[30] A. Kechris. *Classical Descriptive Set Theory.* Graduate Texts in Mathematics 156. Springer-Verlag, New York, NY, 1995.

[31] P. Komjáth. Ordered families of Baire-2-functions. *Real Analysis Exchange*, 15:442–444, 1989-1990.

[32] K. Kunen. *Inaccessibility properties of cardinals.* PhD thesis, Stanford, 1968.

[33] K. Kunen. *Set Theory: an Introduction to Independence Proofs.* Elsevier, New York, NY, 1983.

[34] K. Kuratowski. *Topology, Vol. 1.* Academic Press, New York - London, 1966.

[35] B. Löwe. The length of the full hierarchy of norms. *Rend. Sem. Mat. Univ. Politec. Torino*, 63:161–168, 2005.

[36] H. Mildenberger. Changing cardinal invariants of the reals without changing cardinals or the reals. *Journal of Symbolic Logic*, 63:593–599, 1998.

[37] D. Monk. On general boundedness and dominating cardinals. *Notre Dame J. Formal Logic*, 45:129–146, 2004.

[38] Y. Moschovakis. *Elementary Induction on Abstract Structures.* North Holland Publishing Company, Amsterdam, 1974.

[39] Y. Moschovakis. *Descriptive Set Theory.* American Mathematical Society, 2009.

[40] K. Prikry. Ideals and powers of cardinals. *Bull. Amer. Math. Soc.*, 81:907–909, 1975.

[41] S. Shelah. *Cardinal Arithmetic*. Oxford Logic Guides 29. Oxford University Press, New York, 1994.

[42] R. Solovay. Hyperarithmetically encodable sets. *Trans. Amer. Math. Soc.*, 239:99–122, 1978.

[43] A. Szymański. Some remarks on real-valued measurable cardinals. *Proc. Amer. Math. Soc.*, 104:596–602, 1988.

[44] A. Tarski. Ideale in vollständigen Mengerkörpern I. *Fund. Math.*, 33:51–65, 1945.

[45] S. Thomas. Martin's conjecture and strong ergodicity. *Arch. Math. Logic*, 48:749–759, 2009.

[46] J. Zapletal. Isolating cardinal invariants. *Journal of Mathematical Logic*, 3:143–162, 2003.