DISTRIBUTIVE LAWS FOR BOOLEAN ALGEBRAS

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ABSTRACT. Here I will talk about some classical facts about the distributivity of complete Boolean algebras (including an application of the Covering Lemma). I will also mention some of my theorems in the paper "Weak Distributivity Implying Distributivity".

1. Some Set Theory Definitions

Definition 1.1. X is **transitive** iff

$$(\forall y \in X)(\forall z \in y)\, z \in X.$$

Definition 1.2. A **ctm** is a countable transitive set M such that

$$\langle M, \in \rangle \models \text{ZFC}.$$

Definition 1.3. M is an **inner model** iff M is a proper class such that

1) M is transitive
2) M contains all the ordinals
3) $\langle M, \in \rangle \models$ ZF.

Definition 1.4. A cardinal κ is **singular** iff there is some sequence or ordinals $\langle \kappa_\alpha : \alpha < \lambda \rangle$ such that

$$\kappa = \sup\{\kappa_\alpha : \alpha < \lambda\}$$

where $\lambda < \kappa$ and each $\kappa_\alpha < \kappa$. A cardinal that is not singluar is called **regular**.

$\aleph_0$ is the first infinite cardinal. It is regular.
$\aleph_1$ is the second infinite caridnal. Also regular.
$\aleph_2$ regular. etc.
$\aleph_\omega$ is *singular*.
$\aleph_{\omega+1}$ is regular.
The first incacessible cardinal is regular (all inaccessible cardinals are regular).
The first weakly compact cardinal is regular (all weakly compact cardinals are inaccessible). I may define this later.

The first measurable cardinal is regular (every measurable cardinal is weakly compact). I may define this later.

Those are all the large cardinals that we will talk about.

2. Complete Boolean Algebras

Definition 2.1. A **lattice** is a poset with joins and meets (any two x, y have a least upper bound and a greatest lower bound).

Definition 2.2. A **Boolean algebra** is a lattice with a top (1) and bottom (0) element such that

1) every element has a complement:
$$(\forall x)(\exists y) x \wedge y = 0 \text{ and } x \vee y = 1.$$
2) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Definition 2.3. A **complete Booelan algebra** is a Boolean algebra in which every subset has a least upper bound and a greatest lower bound.

(Write this next definition on another board).

Definition 2.4. Let κ and λ be cardinals. For a Boolean algebra, the (κ, λ)-distributive law states
$$\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} x_{\alpha,\beta} = \bigvee_{f:\kappa\to\lambda} \bigwedge_{\alpha<\kappa} x_{\alpha,f(\alpha)}.$$

Fact: the $\geq$ direction is always true.

We will write (κ, λ)-DL for the (κ, λ)-distributive law.

So the $(\aleph_0, 2)$-DL is analogous to
$$\prod_{n=1}^{\infty} (x_{n,1} + x_{n,2}) = \sum_{f:\mathbb{N}\to 2} \prod_{n=1}^{\infty} x_{n,f(n)}$$
for real numbers.

If $\mathbb{B}$ is a Boolean algebra, then it satisfies the (m, n)-DL for any $m, n \in \mathbb{N}$.

Definition 2.5. Fix a cardinal κ. If $\mathbb{B}$ satisfies the (κ, λ)-DL for every λ, we say $\mathbb{B}$ satisfies the (κ, ∞)-DL.

Definition 2.6. Given a Bolean algebra $\mathbb{B}$, we call $x \in \mathbb{B}$ an **atom** iff $0 < x$ and there is no $y \in \mathbb{B}$ such that $0 < y < x$.

Fact 2.7. *Let $\mathbb{B}$ be a Boolean algebra.*

1) *If every nonzero element of $\mathbb{B}$ is above an atom, then $(\forall \kappa)\, \mathbb{B} \models (\kappa, \infty)$-DL.*

2) *If* $\mathbb{B}$ *has no atoms, then* $(\exists\kappa)\,\mathbb{B} \not\models (\kappa,\infty)$*-DL.*

Fact 2.8. *Let* $\kappa_2 \geq \kappa_1$*. Fix* λ*. For cBa's,*

$$(\kappa_2,\lambda)\text{-}DL \Rightarrow (\kappa_1,\lambda)\text{-}DL.$$

Fact 2.9. *Let* $\lambda_2 \geq \lambda_1$*. Fix* κ*. For cBa's,*

$$(\kappa,\lambda_2)\text{-}DL \Rightarrow (\kappa,\lambda_1)\text{-}DL.$$

3. The Weak Distributive Law

Even if $\mathbb{B}$ does not satisfy the (κ,λ)-DL, it might satisfy the following weaker axiom:

Definition 3.1. Let κ and λ be cardinals. For a Boolean algebra, the (κ,λ)-weak distributive law states

$$\bigwedge_{\alpha<\kappa}\bigvee_{\beta<\lambda} x_{\alpha,\beta} = \bigvee_{f:\kappa\to\lambda}\bigwedge_{\alpha<\kappa}\bigvee_{\beta\leq f(\alpha)} x_{\alpha,\beta}.$$

We will call this the (κ,λ)-WDL.

Proposition 3.2. *Fix* κ *and* λ*. Here are implications between distributive laws for cBa's:*

1) (κ,λ)*-DL* $\Rightarrow$ (κ,λ)*-WDL.*
2) $(\kappa,2)$*-DL* $\Rightarrow$ $(\kappa,2^\kappa)$*-DL* $\Rightarrow$ (κ,κ)*-WDL.*
3) (κ,κ)*-WDL* $\not\Rightarrow$ $(\kappa,2)$*-DL.*
4) *if moreover* $\kappa<\lambda$ *and* λ *is regular, then* (κ,λ)*-WDL is always true.*

Fact 3.3. *If* $\lambda_1 \neq \lambda_2$*, the relaphionship between the* (κ,λ_1)*-WDL and the* (κ,λ_2)*-WDL is more complicated.*

Example 3.4. Let $\mathbb{B} = P(\mathbb{R})$ mod the Lebegsue measure zero sets. Then $\mathbb{B}$ is a cBa such that $\mathbb{B} \models (\aleph_0,\aleph_0)$-WDL but $\mathbb{B} \not\models (\aleph_0,2)$-DL.

Example 3.5. Let $\mathbb{B} = P(\mathbb{R})$ mod the meager sets of reals. Then $\mathbb{B}$ is a cBa such that $\mathbb{B} \not\models (\aleph_0,\aleph_0)$-WDL (and so also $\mathbb{B} \not\models (\aleph_0,2)$-DL).

4. Some Implications

Fact 4.1. *Fix* κ*. Let* $\mathbb{B}$ *be a cBa and let* $\lambda = |\mathbb{B}|$*. Then if* $\mathbb{B} \models (\kappa,\lambda)$*-DL, then* $\mathbb{B} \models (\kappa,\infty)$*-DL.*

Fact 4.2. *Fix* κ*. For cBa's,*

$$(\kappa,2)\text{-}DL \Rightarrow (\kappa,2^\kappa)\text{-}DL.$$

Corollary 4.3. *Let* $\kappa \leq \lambda$*. Then for cBa's,*

$$(\lambda,2)\text{-}DL \Rightarrow (\kappa,\lambda)\text{-}DL.$$

Proof.

$$(\lambda, 2)\text{-DL} \Rightarrow (\lambda, 2^\lambda)\text{-DL} \Rightarrow (\kappa, 2^\lambda)\text{-DL} \Rightarrow (\kappa, \lambda)\text{-DL}$$

□

5. Some Non-Implications

Fact 5.1. *For each infinite cardinal κ, there is a cBa $\mathbb{B}$ such that*

- $(\forall \alpha < \kappa)\, \mathbb{B} \models (\alpha, \infty)$*-DL*
- $\mathbb{B} \not\models (\kappa, 2)$*-DL.*

Solovay asked the question: does the $(\kappa, 2)$-DL imply the (κ, ∞)-DL?

Fact 5.2. *Assuming CH, there is a $\mathbb{B}$ (called Namba forcing) such that*

- $\mathbb{B} \models (\aleph_0, 2)$*-DL.*
- $\mathbb{B} \not\models (\aleph_0, \aleph_2)$*-DL.*

Fact 5.3. *Let κ be a cardinal which is a limit of a countable set of cardinals, and assume $(\forall \mu < \kappa)\, \mu^{\aleph_0} < \kappa$. There is a cBa $\mathbb{B}$ such that:*

- $(\forall \alpha < \kappa)\, \mathbb{B} \models (\aleph_0, \alpha)$*-DL.*
- $\mathbb{B} \not\models (\aleph_0, \kappa)$*-DL.*

So the next question is the following:

Question 5.4. Let $\kappa \geq \aleph_1$. Is there a cBa $\mathbb{B}$ such that $\mathbb{B} \models (\kappa, 2)$-DL but $\mathbb{B} \not\models (\kappa, \infty)$-DL?

Theorem 5.5. *(Main Theorem) Each item in the list below implies the next:*

1) *There is a measurable cardinal.*
2) $(\exists cBa \mathbb{B})\, \mathbb{B} \models (\aleph_1, 2)$*-DL but* $\mathbb{B} \not\models (\aleph_1, \infty)$*-DL.*
3) $(\exists \kappa \geq \aleph_1)(\exists cBa \mathbb{B})\ \mathbb{B} \models (\kappa, 2)$*-DL but* $\mathbb{B} \not\models (\kappa, \infty)$*-DL.*
4) *There is an inner model M such that*

$$M \models ZFC + \exists \text{ a measurable cardinal.}$$

Here is the 1) implies 2) direction:

Theorem 5.6. *(Prikry) Let κ be a measurable cardinal. There is a cBa $\mathbb{B}$ such that $(\forall \alpha < \kappa)\, \mathbb{B} \models (\alpha, 2)$-DL but $\mathbb{B} \not\models (\aleph_0, \kappa)$-DL, and so $\mathbb{B} \not\models (\kappa, 2)$-DL. Hence also, $\mathbb{B} \models (\aleph_1, 2)$-DL but $\mathbb{B} \not\models (\aleph_1, \kappa)$-DL.*

The 2) implies 3) direction is trivial.

We will talk about something close to the 3) implies 4) direction soon.

6. Forcing

How do we prove things about distributive laws?

I might want to say: $\mathbb{B} \models (\kappa, \lambda)$-DL iff whenever you "force" using $\mathbb{B}$, every function from κ to λ in the extension is already in the ground model. $\mathbb{B} \models (\kappa, \lambda)$-WDL iff whenever you "force" using $\mathbb{B}$, every function from κ to λ in the extension is dominated by such a function in the ground model.

Definition 6.1. Given a cBa $\mathbb{B}$, we say $D \subseteq \mathbb{B}$ is **dense** iff

$$(\forall x \in \mathbb{B} - \{0\})(\exists y \in D)\, y \leq x.$$

Definition 6.2. Let M be a ctm. Let $\mathbb{B} \in M$ be such that $M \models$ ($\mathbb{B}$ is a cBa). Then $G \subseteq \mathbb{B}$ is $\mathbb{B}$-generic over M iff for every $D \subseteq \mathbb{B}$ in M, if $M \models$ ($D \subseteq \mathbb{B}$ is dense), then $G \cap D \neq \emptyset$.

Note: if $\mathbb{B} \in M$ has no atoms, then if G is $\mathbb{B}$-generic over M, then $G \notin M$.

Fact 6.3. *Let M be a ctm. Let $\mathbb{B} \in M$ be such that $M \models$ ($\mathbb{B}$ is a ctm). Let G be $\mathbb{B}$-generic over M. Then there is a smallest model $M[G]$ of ZF such that $M \subseteq M[G]$ and $G \in M[G]$. We also have $M[G] \models$ ZFC.*

Theorem 6.4. *Let M be a ctm. Let $\mathbb{B} \in M$ be such that $M \models$ ($\mathbb{B}$ is a ctm). Fix $\kappa, \lambda \in M$. TFAE:*

- *$M \models (\mathbb{B} \models (\kappa, \lambda)$-DL).*
- *For every G that is $\mathbb{B}$-generic over M, we have every $f : \kappa \to \lambda$ in $M[G]$ is already in M.*

Theorem 6.5. *Let M be a ctm. Let $\mathbb{B} \in M$ be such that $M \models$ ($\mathbb{B}$ is a ctm). Fix $\kappa, \lambda \in M$. TFAE:*

- *$M \models (\mathbb{B} \models (\kappa, \lambda)$-WDL).*
- *For every G that is $\mathbb{B}$-generic over M, for every $f : \kappa \to \lambda$ in $M[G]$, there is some $g : \kappa \to \lambda$ in M such that*

$$(\forall \alpha < \kappa)\, f(\alpha) \leq g(\alpha).$$

7. The Core Model

(Only do this if have time).

Theorem 7.1. *(Dodd-Jensen) Assume there is no inner model with a measurable cardinal. Then in any forcing extension of V, there is no inner model with a measurable cardinal. Moreover, there is an inner model K such that*

1) *K is computed the same way in every forcing extension*

2) *In any forcing extension of V the following holds: for every cardinal λ, we have λ is regular iff $K \models \lambda$ is regular.*

Corollary 7.2. *Suppose there is a cBa $\mathbb{B}$ and an inacessible cardinal κ (so κ is a regular limit cardinal) such that*

1) $(\forall \alpha < \kappa)\, \mathbb{B} \models (\alpha, 2)$-*DL.*
2) $\mathbb{B} \not\models (\aleph_0, \kappa)$-*DL.*

Then there is an inner model with a measurable cardinal.

Proof. Assume there is no inner model with a measurable cardinal.

Let G be $\mathbb{B}$-generic over V. By 1), the cardinals below κ in V are the same as the cardinals below κ in $V[G]$. So κ is a limit of cardinals in $V[G]$, so κ is cardinal in $V[G]$.

But by 1) and 2) together, there must be a function $f : \aleph_0 \to \kappa$ in $V[G]$ whose range is cofinal in κ. Hence κ is singular in $V[G]$. So κ is singular in $K^{V[G]} = K$.

On the other hand, because κ is regular in V, κ is regular in K. This is impossible. □

8. Weak Distributivity

Here is one of my theorems:

Theorem 8.1. *(H.) Let λ be a cardinal and let $\mathbb{B}$ be a cBa. Let $\mu = \lambda^{\aleph_0}$. Then if $\mathbb{B} \models (\mu, \aleph_0)$-WDL, then $\mathbb{B} \models (\lambda, 2)$-DL.*

Corollary 8.2. *Let $\mathbb{B}$ be a forcing such that every function from $2^{\aleph_0}$ to $\aleph_0$ in the extension is dominated by such a function in the ground model. Then every real number in the extension is in the ground model.*

The following is easy:

Fact 8.3. *There is a cBa $\mathbb{B}$ such that $(\forall \mu)\, \mathbb{B} \models (\mu, \aleph_1)$-WDL but $\mathbb{B} \not\models (\aleph_0, 2)$-DL.*

So there is a qualitative difference between $\aleph_0$ and $\aleph_1$.

The property of $\aleph_0$ we used was similar to weak compactness.

Theorem 8.4. *(H.) Let κ be a weakly compact cardinal. Let $\mathbb{B}$ be a cBa such that*

1) $(\forall \alpha < \kappa)\, \mathbb{B} \models (\alpha, 2)$-*DL.*
2) $\mathbb{B} \models (2^\kappa, \kappa)$-*WDL.*

Then $\mathbb{B} \models (\kappa, 2)$-DL.

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