A THEOREM OF LAJOS SOUKUP THAT EVERY FAMILY OF FUNCTIONS OF SIZE ω_1 HAS A BOUNDED SUBFAMILY OF SIZE ω

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Abstract. Here we communicate the following theorem of Lajos Soukup: Let $\mathcal{F}$ be a size ω_1 family of functions from ω to ω. Then there is a size ω subfamily all of whose members are everywhere dominated by a single function.

1. Introduction

One question in combinatoral set theory is whether there can exist a large family of objects such that all subfamilies of a certain size have a given property. At the 7th Young Set Theory Workshop in 2014 in Będlewo, Lajos Soukup gave a tutorial related to this topic. In private communication he explained Theorem 3.1. With his permission it is reproduced here.

2. A Theorem of Todorcevic

The main theorem uses the following theorem of Todorcevic. This comes from Todorcevic's theory of Oscillations [3] (reference TBD).

Theorem 2.1 (Todorcevic)**.** *Let $X \subseteq {}^\omega\omega$. Assume that X is $\le^*$-unbounded. Also assume that it is countably $\le^*$-directed, meaning*

$$(\forall Y \in [X]^\omega)(\exists g \in X)(\forall f \in Y)\, f \le^* g.$$

Then there are $f_1, f_2 \in X$ such that

$$f_1 \le f_2.$$

3. Main Theorem

Theorem 3.1 (Main Theorem)**.** *Let $\mathcal{F}$ be a size ω_1 family of functions from ω to ω. Then there is a size ω subfamily $\mathcal{F}' \subseteq \mathcal{F}$ such that there is a single function $g : \omega \to \omega$ which everywhere dominates every function in $\mathcal{F}'$.*

Proof. There are two cases. The first case is that $\mathfrak{b} > \omega_1$. For more on the *bounding number* $\mathfrak{b}$, see [1]. There must exist a $g : \omega \to \omega$ which $\le^*$-dominates (eventually dominates) $\mathcal{F}$. So by the pigeonhole principle, there exists a finite modification g' of g which $\le$-dominates ω_1 many members of $\mathcal{F}$ (for each $f \in \mathcal{F}$, a finite modification of g $\le$-dominates f, and there are only ω many finite modifications of g). In particular, g' $\le$-dominates ω members of $\mathcal{F}$, and we are done.

The other case is that $\mathfrak{b} = \omega_1$. Fix an enumeration

$$\mathcal{F} = \{f_\alpha : \alpha < \omega_1\}.$$

By possibly increasing the functions in $\mathcal{F}$, get a new family

$$\mathcal{F}' = \{f'_\alpha : \alpha < \omega_1\}$$

such that

$$(\forall \alpha < \beta < \omega_1)\, f'_\alpha <^* f'_\beta$$

(to do this, for each f_β pick a f'_β which is the max of f_β and a function h which everywhere dominates $X := \{f'_\alpha : \alpha < \beta\}$ but is not everywhere dominated by any function in X. Hence if $\alpha < \beta$, then $f_\beta \not\le f_\alpha$. Also assume that $\mathcal{F}'$ is $\le^*$-unbounded (to do this, replace each f'_α in $\mathcal{F}'$ with the max of itself and h_α, where $\{h_\alpha : \alpha < \omega_1\}$ is an $\le^*$-unbounded family).

Now $\mathcal{F}'$ is $\le^*$-unbounded and countably directed with respect to $\le^*$. In fact every size ω_1 subfamily of $\mathcal{F}'$. of it is $\le^*$-unbounded and is countably directed with respect to $\le^*$.

Now let $c : [\omega_1]^2 \to 2$ be the coloring defined as follows:

$$c(\{\alpha,\beta\}) := \begin{cases} 0 & \text{if } f_\alpha \not\le f_\beta \\ 1 & \text{if } f_\alpha \le f_\beta \end{cases}.$$

By the partition relation $\omega_1 \to (\omega_1, \omega{+}1)$ (the Dushnik-Miller Theorem [2]), we have two cases:

<u>Case 1</u>:

$$(\exists X \in [\mathcal{F}']^{\omega_1})(\forall \{\alpha,\beta\} \in [X]^2)\, c(\{\alpha,\beta\}) = 0.$$

Fix such an $X \in [\mathcal{F}']$. Now X is $\le^*$-unbounded and countably directed with respect to $\le^*$. By Theorem 2.1, there are $f_\alpha, f_\beta \in \mathcal{F}'$ with $\alpha \neq \beta$ such that $f_\alpha \le f_\beta$. It must be that $\alpha < \beta$. So we have $c(\{\alpha,\beta\}) = 1$, which is a contradiction.

<u>Case 2</u>:

$$(\exists X \in [\mathcal{F}]^{\omega+1})(\forall \{\alpha,\beta\} \in [X]^2)\, c(\{\alpha,\beta\}) = 1$$

Fix such an $X \in [\mathcal{F}']$. Let X be

$$X = \{f'_{\alpha_1}, f'_{\alpha_2}, ..., f'_{\alpha_\omega}\}$$

with

$$\alpha_1 < \alpha_2 < ... < \alpha_\omega.$$

We have

$$f'_{\alpha_1} \leq f'_{\alpha_2} \leq ... \leq f'_{\alpha_\omega}$$

That is, f'_{α_ω} witnesses that the first ω members of X are $\leq$-bounded. Hence, $f'_{\alpha,\omega}$ witnesses that the set

$$Y := \{f_{\alpha_n} : n < \omega\}$$

is $\leq$-bounded, which is what we wanted to show. □

4. QUESTIONS

Question 4.1. Let λ, κ be cardinals. with $\kappa \leq \lambda$ and κ regular. For each cardinal μ, what is the smallest cardinal $F(\lambda, \kappa, \mu)$ such that every family of that size of functions from λ to κ has a subfamily of size μ that is $\leq$-bounded?

REFERENCES

[1] Blass, A., 2010. *Combinatorial cardinal characteristics of the continuum*, in Foreman, M. and Kanamori A. (Eds.) Handbook of Set Theory Volume 1, Springer, New York, pp. 395-489.

[2] Dushnik, B., Miller, W., 1941. *Partially ordered sets*. Amer. J. Math. 63, pp 600-610.

[3] Todorcevic, S., 1998. *Oscillations of Sets of Integers*. Advances in Applied Mathematics 20, pp 220-252.

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