A RAMSEY THEOREM AT A MEASURABLE CARDINAL

DAN HATHAWAY

1. Background

I originally thought Shelah had proved the Galvin-Prikry theorem at a Ramey cardinal because of the following math overflow post:

https://mathoverflow.net/questions/132665/why-does-the-generalised-galvin-prikry-theorem-only-hold-at-ramsey-cardinals

Let κ be a cardinal. Call the **metric topology** on $[\kappa]^\omega$ the topology generated by sets of the form $[s]$ for $s \in [\kappa]^{<\omega}$, where

$$[s] := \{A \in [\kappa]^\omega : s \text{ is an inital segment of A}\}.$$

Call $\mathcal{X} \subseteq [\kappa]^\omega$ **Borel** iff it is in the smallest σ-algebra generated from the open sets (in the metric topology). Call $f : [\kappa]^\omega \to 2$ clopen iff $f^{-1}(0)$ and $f^{-1}(1)$ are both clopen (in the metric topology). Call $f : [\kappa]^\omega \to 2$ Borel iff $f^{-1}(0)$ and $f^{-1}(1)$ are both Borel (in the metric topology). Set $f : [\kappa]^\omega$ is Ramsey iff there is a set $H \in [\kappa]^\kappa$ such that $|f``[H]^\omega| = 1$.

I thought Shelah proved that if κ is a Ramsey cardinal, then if $f : [\kappa]^\omega \to 2$ is Borel, then f is Ramsey. However when looking through the paper, I cannot find this.

The mathoverflow poster seems to think the "Borel" case follows from "every κ-block has a κ-barrier", but now I think maybe they are misquoting what the Galvin-Prikry theorem is. At the very least, they seem to think that "every κ-block has a κ-barrier" implies every clopen $f : [\kappa]^\omega \to 2$ is Ramsey. And Shelah does claim that "every κ-block has a κ-barrier" if κ is Ramsey (see the introduction of his paper).

I haven't been able to figure out the paper, but I do think I figured out how to show that if κ is a measurable cardinal, then every clopen $f : [\kappa]^\omega \to 2$ is Ramsey. The proof is in the next section

2. The Clopen Theorem

This theorem is a modification of the "ultrafilter proof" of the Nash-Williams theorem that every clopen $f : [\omega]^\omega \to 2$ is Ramsey. Andreas Blass told me this proof. Maybe he used that $\mathcal{U}$ was a P-Point?

Theorem 2.1. *Let κ be a measurable cardinal. Let $\mathcal{U}$ be a normal ultrafilter on κ. Let $T \subseteq {}^{<\omega}\kappa$ be a wellfounded tree such that the nodes are strictly increasing functions. Let $c' : \mathit{LeafNodes}(T) \to 2$. Let $c : [\kappa]^\omega \to 2$ be the induced coloring where $(\forall x \in [\kappa]^\omega)\, c(x) := c'(s)$ where $s \in [\kappa]^{<\omega}$ is the unique initial segment of x that is in $\mathit{LeafNodes}(T)$. Then there is a set $H \in \mathcal{U}$ such that $|c``[H]^\omega| = 1$.*

Proof. The idea is to first 1) assign a color $c'(s)$ to *every* node $s \in T$ and 2) pick a set $A_s \in \mathcal{U}$ for every non-leaf node $s \in T$. This should be done so that for each non-leaf node $s \in T$, each child $s^\frown\alpha$ for $\alpha \in A_s$ has the same color as s. We can easily do this by induction on rank: we do nothing with the nodes of rank 0 (the leaf-nodes), then we deal with the nodes of rank 1, etc. The color of the root node will be the color that we homogonize to.

We now need to somehow take a diagonal intersection of all the A_s's to get our H. Let $H \subseteq \kappa$ be the set of all limit ordinals $\alpha < \kappa$ such that

$$(\forall \beta < \alpha)\, \alpha \in \bigcap \{A_s : s \in {}^{<\omega}\beta\}.$$

We have $H \in \mathcal{U}$ because it is a diagonal intersection of sets in $\mathcal{U}$. We claim that for each $s \in T$ and $\alpha > \max(s)$, if $\alpha \in H$ then $\alpha \in A_s$. This is because if $\alpha \in H$, then α is a limit ordinal, so we can pick $\beta < \alpha$ such that $\max(s) < \beta$. So then A_s will be in $\{A_t : t \in {}^{<\omega}\beta\}$. Etc.

One can check that this H works.

□

Corollary 2.2. *Let κ be a measurable cardinal. Let $\mathcal{U}$ be a normal ultrafilter on κ. Let $f : [\kappa]^\omega \to 2$ be clopen (in the metric topology). Then there is a set $H \in \mathcal{U}$ such that $|f``[H]^\omega| = 1$.*

3. The Borel Theorem

Observation 3.1. Let κ be measurable and $\mathcal{U}$ be a normal ultrafilter on κ. For each $n \in \omega$ let $f_n : [\kappa]^\omega \to 2$ be homogeneous on a set $A_n \in \mathcal{U}$. Let $f = \lim_{n\to\omega} f_n$ (so for each $x \in [\kappa]^\omega$, $\lim_{n\to\omega} f_n(x)$ exists). Then f is homogeneous on $\bigcap_{n<\omega} A_n \in \mathcal{U}$.

Fact 3.2. *Let κ be a measurable cardinal. Let $f : [\kappa]^\omega \to 2$ be open, meaning $f^{-1}(0)$ is open. Then $f = \lim_{n\to\omega} f_n$ where each $f_n : [\kappa]^\omega \to 2$ is clopen.*

Proof. Let $f : [\kappa]^\omega \to 2$ be open. Let $S \subseteq [\kappa]^{<\omega}$ be such that $f(x) = 1$ iff $x \in [s]$ for some $s \in S$. For each n, let $f_n : [\kappa]^\omega \to 2$ be the function $f_n(x) = 1$ iff $x \sqsupseteq s$ for some $s \in S \cap [\kappa]^{\leq n}$. One can check that each f_n is clopen, and that $\lim_{n\to\omega} f_n = f$. □

Note: the ideas here can perhaps simplfiy the clopen theorem in the last section.

Definition 3.3. The collection of **Baire** functions from $[\kappa]^\omega$ to 2 is the smallest collection of functions such that

- if $f : [\kappa]^\omega \to 2$ is clopen, then f is Baire,
- if $f = \lim_{n\to\omega} f_n$ where each f_n is Baire, then f is Baire.

Note: Borel should be the same thing as Baire.

Corollary 3.4. *Let κ be measurable cardinal. Let $\mathcal{U}$ be a normal ultrafilter on κ. Suppose $f : [\kappa]^\omega \to 2$ is Baire. Then there is a $H \in \mathcal{U}$ such that $|f``[H]^\omega| = 1$.*

Note: if κ is measurable and $\gamma < \kappa$, we can probably extend these ideas to functions which are γ-limits of functions which can be homogeneized by sets in the normal ultrafilter.

References

[1] E. Ellentuck. A New Proof that Analytic Sets are Ramsey. J. Symbolic Logic 39 (1974), no 1, 163-165.

[2] F. Galvin and K. Prikry. Borel sets and Ramsey's theorem. J. Symbolic Logic 38 (1973), no 2, 193-198.

[3] A. Kanamori. The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings. Berlin: Springer, 2009.

[4] C. Nash-Williams. On Well-quasi-ordering transfinite sequences. Proceedings of the Cambridge Philosophical Society, 1965, no 61 (1), 33-39.

[5] S. Shelah. Better Quasi-orders for Uncountable Cardinals. Israel J. Math. (1982) 42:177.

[6] J. Silver. Every Analytic Set is Ramsey. J. Symbolic Logic 35 (1970), no 1, 60-64.

Mathematics Department, University of Denver, Denver, CO 80208, U.S.A.

Email address: `Daniel.Hathaway@du.edu`