DOMINATING A FUNCTION THAT BLOWS UP

DAN HATHAWAY

Abstract. Fix a real $r \in \mathbb{R}$. We show that if $g : \mathbb{R} \to \mathbb{R}$ everywhere dominates the function $f(x) = (x - r)^{-2}$, then $r \in L[g]$. That is, r is constructible from the graph of g. This is taken from the author's PhD Thesis (Proposition V.14).

1. Main theorem

Theorem 1.1. *Let M be a transitive model of ZF. Let $f : \mathbb{R} \to \mathbb{R}$ and $r \in \mathbb{R}$ be such that for each $n \in \mathbb{R}$, there is a neighborhood U of r satisfying*

$$(\forall x \in U \cap M - \{r\})\, n \leq f(x).$$

Let $g : \mathbb{R}^M \to \mathbb{R}$ be in M and suppose

$$(\forall x \in \mathbb{R}^M)\, f(x) \leq g(x).$$

Then $r \in M$.

Proof. Fix f and g as in the statement of the theorem. Let $\langle W, \prec \rangle$ be the poset defined as follows:

1) Each element of W is a pair (C, n) such that C is a closed subinterval of $\mathbb{R}$ and $n \in \omega$. Furthermore,
$$(\forall x \in C)\, n \leq g(x)$$
2) If $(C_1, n_1), (C_2, n_2) \in W$, then $(C_2, n_2) \prec (C_1, n_1)$ iff
$$C_2 \subseteq C_1 \text{ and } n_2 > n_1.$$

Note that $\langle W, \prec \rangle^M$ is well-founded, because if not, then within M there would be an infinite sequence $(C_0, n_0) \succ (C_1, n_1) \succ ...$, and letting $y \in \bigcap_n C_n$, we would have $(\forall n \in \omega)\, n \leq g(y)$ which is impossible.

Since $\langle W, \prec \rangle^M$ is well-founded in M and well-foundedness is absolute, it is well-founded in V. That is, $\langle W, \prec \rangle^M$ has no infinite paths in V.

Now, assume towards a contradiction, that $r \notin M$. We will construct an infinite path through $\langle W, \prec \rangle^M$ (in V). Assume that we have the path

$$(C_n, n) \prec (C_{n-1}, n - 1) \prec ... \prec (C_0, 0)$$

where the elements of the path are in W^M, and assume $r \in C_n$ We will find a C_{n+1} such that $(C_{n+1}, n+1) \prec (C_n, n)$ and $r \in C_{n+1}$. Repeating this procedure will give us an infinite path through $\langle W, \prec \rangle$, which is a contradiction.

Since $r \notin M$, there is some neighborhood U of r such that

$$(\forall x \in U \cap M)\, n + 1 \leq f(x).$$

Since U is a neighborhood containing r, fix a closed interval $\tilde{C} \subseteq U$ of M containing r. Since $\tilde{C}$ and C_n are two closed intervals of M containing r, define $C_{n+1} := \tilde{C} \cap C_n$. Now C_{n+1} is also a closed interval of M and it satisfies

- $C_{n+1} \subseteq U$
- $r \in C_{n+1}$
- $C_{n+1} \subseteq C_n$.

Now

$$(\forall x \in C_{n+1})\, n + 1 \leq f(x) \leq g(x).$$

Hence, $(C_{n+1}, n + 1)$ is in W^M. This is what we wanted to show. □

Corollary 1.2. *Fix a real $r \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be the function*

$$f(x) := \begin{cases} \frac{1}{(x-r)^2} & \text{if } x \neq r, \\ 0 & \text{if } x = r. \end{cases}$$

Let $g : \mathbb{R} \to \mathbb{R}$ everywhere dominate f. Then $r \in L[g]$. That is, r is constructible from the graph of g.

References

[1] Blass, A., 2010. *Combinatorial cardinal characteristics of the continuum*, in Foreman, M. and Kanamori A. (Eds.) Handbook of Set Theory Volume 1, Springer, New York, pp. 395-489.

[2] Hathaway, D., 2015. *Generalized Domination.* University of Michigan. Phd dissertation.
http://www.danthemanhathaway.com/ProfLife/NotesAndOther/index.html

[3] Jockusch, C., 1968. *Uniformly introreducible sets.* J. Symbolic Logic, Volume 33, pp. 521536.

Department of Mathematics, University of Vermont, Innovation Hall, 82 University Place, Burlington, VT 05405 U.S.A.

E-mail address: `Daniel.Hathaway@uvm.edu`

URL: `http://www.danthemanhathaway.com/ProfLife/index.html`