Summary of “Real Analysis” by Royden

Dan Hathaway

May 2010

This document is a summary of the theorems and definitions and theorems from Part 1 of the book “Real Analysis” by Royden. In some areas, such as Set Theory, I have not included the simple results that almost every mathematitions knows off the top of their head.

1 Chapter 1: Set Theory

Royden assumes the notion of ordered pairs are given apriori in set theory (as opposed to the traditional use of the Kuratowski definition). Functions are defined in terms of pairs. Sequences are functions with either $\mathbb{N}$ or $\{1, ..., n\}$ as a domain. He calls finite sequences *n-tuples*. He also defines monotone as meaning strict.

Definition 1.1. *The* ***domain*** *of a function f is $\{x : \exists_y f(x) = y\}$ The* ***range*** *of a function f is $\{y : \exists_x f(x) = y\}$. $F[A] = \{y : \exists_{x \in A} f(x) = y\}$. $F^{-1}[A] = \{x : \exists_{y \in A} f(x) = y\}$.*

Theorem 1.1. *$f[\bigcup_i A_i] = \bigcup_i f[A_i]$, $f[\bigcap_i A_i] \subseteq \bigcap_i f[A_i]$, $f^{-1}[\bigcup_i A_i] = \bigcup_i f^{-1}[A_i]$, and $f^{-1}[\bigcap_i A_i] = \bigcap_i f^{-1}[A_i]$. If f has domain X and $A \subseteq X$ then $f^{-1}[X \backslash A] = X \backslash f^{-1}[A]$.*

Theorem 1.2. ***Hausdorff Maximal Principle****: Let $<$ be a partial ordering on a set X. There exists a subset S of X that is is linearly ordered by $<$, and S is a maximal set with this property.*

2 Chapter 2: The Real Number System

Definition 2.1. ***Field Axioms****: The usual field axioms.*

Definition 2.2. ***Axioms of Order****: There is a subset P of $\mathbb{R}$ that satisfies the following:*

1) $x, y \in P \Rightarrow x + y \in P$.

2) $x, y \in P \Rightarrow xy \in P$.

3) $x \in P \Rightarrow -x \notin P$.

4) $x \in R \Rightarrow x = 0 \vee x \in P \vee -x \in P$

Definition 2.3. ***Completeness Axiom****: Every nonempty subset of R that has an upper bound has a least upper bound.*

It should follow that $\mathbb{R}$ is the only model up to isomorphism of the above three classes of axioms (any two complete ordered sets that contain the rationals as a dense subset are order isomorphic [Jech]). More specifically, if A and B have dense subsets that are order isomorphic, there there is one and only one extension of this isomorphism to maps from A to B. The result follows from the fact that a field isomorphism from A to B can map the prime subfied of A to the prime subfield of B in one and only one way.

Theorem 2.1. *Any field that satisfies the order axioms contains sets isomorphic to the natural numbers, the integers, and the rational numbers.*

Theorem 2.2. *Given any $x \in \mathbb{R}$ there is an integer n s.t. $x < n$.*

I found the following is a useful way to think about limsup and liminf:

Proposition 2.1. $\limsup_{n\to\infty} x_n$ *is the least and* $\liminf_{n\to\infty} x_n$ *is the greatest cluster point of* $\{x_n\}_{n=1}^{\infty}$

Theorem 2.3. ***Lindelöf's Theorem:*** *Let* C *be a collection of open subsets of* R. *There is a countable subcollection* $\{O_i\}_{i=1}^{\infty}$ *of* C *such that* $\bigcup_{O\in C} O = \bigcup_{i=1}^{\infty} O_i$.

Here is a little research I did online: A topological space is **Lindelöf** if the above theorem holds, **second countable** if its topology has a countable basis, and **seperable** if it has a countable sense subset. In a metric space, these three conditions are equivalent. Second countable implies the other two conditions. There exists a space that is **Lindelöf** but not seperable: $(I \times I)_{lex}$ (the unit square with the lexographic order topology). There exists a space that is **Lindelöf** and seperable but not second countable: $\mathbb{R}_l$ ($\mathbb{R}$ with the lower limit topology). There exists a space that is seperable but not Lindelöf: the plane with the topology where the basic open sets around a point a have the form $(B_\epsilon - L_1 \cup ... \cup L_k) \cup a$ where each L_i is a line through a.

Theorem 2.4. ***Heine-Borel:*** *A closed and bounded subset of* R *is compact.*

The Heine-Borel theorem is equivalent to the following:

Theorem 2.5. *If* C *is a collection of closed sets such that the intersection of every finite subcollection is nonempty, then the intersection of the entier collection is nonempty.*

Theorem 2.6. ***Uniform Convergence Theorem:*** *If* $(f_n)_n$ *is a sequence if continuous functions defined on a compact set that converge uniformly to* f *then* f *is continuous.*

What wikipedia says about derivitives of functions that converge uniformly:

Theorem 2.7. *If* f_n *converges uniformly to* f, *and if all the* f_n *are differential, and the derivitives converge uniformly to* g, *then* f *is differentiable and its derivative is* g.

Proposition 2.2. *The set of points at which a function is continuous is* G_δ *(the countable intersection of open sets).*

3 Chapter 3: Lebesgue Measure

3.1 Bootstrap Definitions

There cannot be a function m such that the following four conditions hold:

1) m is defined on all of $P(\mathbb{R})$.

2) For an interval I, $mI = l(I)$.

3) For a sequence $\{E_n\}_{n=1}^{\infty}$ of disjoint sets (for which m is defined) $m(\bigcup_n E_n) = \sum mE_n$.

4) $m\{x + y : x \in E\} = mE$ for all $y \in \mathbb{R}$, $E \subseteq \mathbb{R}$.

We will drop the first condition (but will want the function to be defined on a σ-algebra). Note: Alternatively, we could have changed condition 3 to either "finite additivity" or "countable subadditivity" as is done with outer measure.

Definition 3.1. $m^*A = \inf_{A\subseteq(\bigcup_n I_n)} \sum_n l(I_n)$ *(each* I_n *is an interval). This is called the **outer measure**.*

Proposition 3.1. *The outer measure of an interval is its length. (The (a,b) case follows from the [a,b] case).*

Proposition 3.2. $m^*(\bigcup_n A_n) \le \sum_n m^*A_n$.

Proposition 3.3. m^* *is translation invariant.*

Definition 3.2. *A set* E *is **measurable** if* $m^*A = m^*(A\cap E) + m^*(A\backslash E)$ *for all* $A \subseteq \mathbb{R}$. *If* E *is measurable, then we write* mE *instead of* m^*E.

Note: If E is measurable then $\mathbb{R}\backslash E$ is measurable by definition.

Lemma 3.1. *If* $m^*E = 0$, *then* E *is measurable.*

Lemma 3.2. *If $\{E_n\}_{n=1}^{\infty}$ are pairwise disjoint measurable sets then $m(\bigcup_n E_n) = \sum_n mE_n$.*

We have therefore shown the following:

Theorem 3.1. *The function m^* defined on the measurable subsets of $\mathbb{R}$ satisfies conditions 2, 3, and 4.*

We can also show that the measurable sets are closed under countable union, which gives us the following:

Theorem 3.2. *The family of measurable subsets of $\mathbb{R}$ is σ-algebra.*

We can also show that (a, ∞) is measurable for any $a \in \mathbb{R}$. The Borel sets result from the intersection of every σ-algebra that contains the sets of the form (a, ∞), so every Borel set is measurable.

The measure function aslo satisfies the following, which we might as well place here:

Proposition 3.4. *If $E_1 \supseteq E_2 \supseteq \ldots$ is a sequence of measurable subsets of $\mathbb{R}$ then $m(\bigcap_n E_n) = \lim_{n\to\infty} mE_n$.*

Definition 3.3. *A property holds a.e. (almost everywhere) if the set of points at which it does not hold has measure zero.*

3.2 Definition of Measurable Function

Definition 3.4. *$f : \mathbb{R} -> \mathbb{R}^*$ is a **measurable function** if $\{x : f(x) > \alpha\}$ is measurable for all $\alpha \in \mathbb{R}$.*

The above definition is equivalent to that which we could get by replacing $<$ with either $\leq$, $>$, or $\geq$. Note that we did not need to include the fact that f has a measurable domain in the definition becaues this is implied by the fact that $\cup_{n=1}^{\infty} f^{-1}[(-n, \infty)] = f^{-1}[\cup_{n=1}^{\infty}(-n, \infty)] = f^{-1}[\mathbb{R}]$ is measurable.

Proposition 3.5. *If f is measurable then $\{x : f(x) = \alpha\}$ is measurable for all $\alpha \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$.*

Proposition 3.6. *If f is measurable and B is Borel then $f^{-1}[B]$ is measurable.*

Theorem 3.3. *If f and g and measurable then so is $\alpha f + \beta g$. The* sup *and* inf *of sequences of measurable functions are also measurable.*

Proposition 3.7. *If $f = g$ a.e. and f is measurable then g is measurable.*

It follows from the above two propositions that if f_n approaches f a.e. and each f_n is measurable then f is measurable.

3.3 Big Theorems

3.3.1 Littlewood's First Principle

Every measurable set is nearly a finite sum of intervals.

Theorem 3.4. *The following are equivalent:*

1) E is measurable.

2) Given $\epsilon > 0$, there is an open set $O \supset E$ with $m^*(O \backslash E) < \epsilon$.

3) Given $\epsilon > 0$, there is a G_δ set $G \supset E$ with $m^*(G \backslash E) = 0$.

The analogue of 2) and 3) for closed sets and F_σ, respectively, hold. Note that by 3) a measurable set E is determined by a G_δ set (G) and a set of measure zero ($E \backslash G$). Thus, the cardinality of the collection of measurable sets is the same as the maximum of $2^{\aleph_0}$ (the number of G_δ sets) and the cardinality of the collection of sets of measure zero. Since there are at least $2^{\aleph_0}$ many sets of measure zero, we have that the cardinality of collection of measurable sets is the same as the cardinality of the collection of sets with measure zero.

3.3.2 Littlewood's Second Principle

Every measurable function is nearly continuous.

Proposition 3.8. *Let f be a measurable, defined on $[a, b]$, taking on the values $\pm\infty$ on a set of measure zero. Then given $\epsilon > 0$ we can find a step function s and a continuous function c such such that $|f - s| < \epsilon$ and $|f - c| < \epsilon$ except on a set of measure less than ϵ.*

Theorem 3.5. ***Lusin's Theorem:*** *Let f be a measurable function on the interval $[a, b]$. Then given $\delta > 0$, there is a continuous function g on $[a, b]$ such that $m\{x : f(x) \neq g(x)\} < \delta$.*

Littlewood's second principle can be talked about in the context of L^p functions.

3.3.3 Littlewood's Third Principle

Every convergent sequence of measurable functions is nearly uniformly convergent.

Theorem 3.6. *Let E be measurable with finite measure, and $\{f_n\}_{n=1}^{\infty}$ a sequence of measurable functions defined on E that converge pointwise to a function f. Then given $\epsilon > 0$ and $\delta > 0$, there is a measurable set $A \subseteq E$ with $mA < \delta$ and an N such that $|f_n(x) - f(x)| < \epsilon$ for all $x \notin A$ and $n \geq N$.*

Since the set E above was any measurable set with finite measure, we can modify the result to apply for sequences of measurable functions that converge a.e. on E instead of everywhere on E. The following is even more general:

Theorem 3.7. ***Egoroff's Theorem:*** *Let E be measurable with finite measure, and $\{f_n\}_{n=1}^{\infty}$ a sequence of measurable functions defined on E that converge a.e to a function f. Then given $\delta > 0$, there is a measurable set $A \subseteq E$ such that $mA < \delta$ and f_n converges uniformly to f on $E \backslash A$.*

4 Chapter 4: The Legesgue Integral

4.1 Simple Functions

Definition 4.1. *A function f is called **simple** if it can be written in the form $\sum_{i=1}^{n} a_i \chi_{E_i}(x)$ where each E_i is a measurable set. If f vanishes outside a set of finite measure, we define its integral $\int f$ to be $\sum_{i=1}^{n} a_i mE_i$.*

We will adomt the principle that whenever we have defined an integra $\int f$, if E is measurable then $\int_E f$ denotes $\int (f \cdot \chi_E)$.

Lemma 4.1. *Let ϕ and ψ be simple functions which vanish ouside a set of finite measure. Then $\int (a\phi + b\psi) = a \int \phi + b \int \psi$. If $\phi \geq \psi$ a.e. then $\int \phi \geq \int \psi$.*

4.2 Defining Integral of Bounded Function on Finite Domain

Over the next several sections we will be defining the Lebesgue Integral for larger and larger classes of (measurable) functions. The measurable requirement comes from the following:

Proposition 4.1. *Let f be bounded with domain some measurable set E with finite measure. Then f is measurable iff $\inf_{f \leq \psi} \int_E \psi = \sup_{f \geq \phi} \int_E \phi$ where ψ and ϕ range over all simple functions.*

Definition 4.2. *If f is a bounded measurable function with a domain of finite measure then its **Lebesgue Integral** $\int f$ is defined as $\inf_{f \leq \psi} \int_E \psi$. where ψ ranges over all simple functions.*

Theorem 4.1. ***Bounded Convergence Theorem:*** *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions with domain E where mE finite. Suppose there is some M such that $|f_n(x)| < M$ for all x and n. If there is a function f with domain E such that $f_n(x) \to f(x)$ for all x, then $\int f = \lim_{n\to\infty} \int f_n$.*

The Bounded Convergence Theorem is implied by the Lebesgue convergence theorem which we will state shortly.

4.3 Defining Integral of Nonnegative Function

Definition 4.3. *If f is a nonnegative measurable function defined on a measurable set E then $\int f$ is defined as $\sup_{h \leq f} \int_E h$ where h ranges over all bounded measurable functions such that $m\{x : h(x) \neq 0\}$ is finite.*

Only when the integral is finite do we call the function integrable:

Definition 4.4. *A nonnegative measurable function f is called integrable over the measurable set E iff $\int_E f < \infty$.*

Proposition 4.2. *Let f and g be nonnegative functions. If f is integrable over E and g(x) ¡ f(x) on E, then g is integrable over E.*

Theorem 4.2. ***Fatou's Lemma**: If $\{f_n\}_{n=1}^{\infty}$ is a sequence of nonnegative measurable functions with domain E and f is a function with domain E such that $f_n(x) \to f(x)$ a.e. on E, then $\int f \leq \liminf_n \int f_n$.*

Theorem 4.3. ***Monotone Convergence Theorem**: Let $\{f_n\}_{n=1}^{\infty}$ be an increasing sequence of nonnegative measurable functions with domain E, and let $f = \lim_n f_n$ a.e. Then $\int f = \lim_n \int f_n$.*

In the above theorem, remember that the *sup* (or inf) of a sequence of measurable functions is measurable. The Monotone Convergence Theorem gives us the following two useful corollaries:

Corollary 4.1. *If u_n is a sequence of nonnegative measurable functions and $f = \sum_{n=1}^{\infty} u_n$ then $\int f = \sum_{n=1}^{\infty} \int u_n$.*

Corollary 4.2. *Let f be a nonnegative measurable function with domain $\bigcup_n E_n$ where $\{E_n\}_{n=1}^{\infty}$ is a sequence of disjoint measurable sets. Then $\int f = \sum_n \int_{E_n} f$.*

4.4 Defining General Lebesgue Integral

Definition 4.5. *A measurable function f is said to be integrable over a measurable set E iff both its positive f^+ and negative f^- parts are integrable over E. If f is integrable over E we define $\int_E f = \int_E f^+ - \int_E f^-$.*

Theorem 4.4. ***Lebesgue Convergence Theorem**: Let g be integrable over E and let f_n be a sequence of measurable functions such that $|f_n| \leq g$ on E. Suppose we have $f(x) = \lim_n f_n(x)$ for almost all $x \in E$. Then $\int_E f = \lim_n \int_E f_n$.*

Proposition 4.3. *Let f and g be measurable functions that are integrable. We have:*

1) $\int (af + bg) = a \int f + b \int g$.

2) If $f \leq g$ a.e. then $\int f \leq \int g$.

3) If $f = g$ a.e. then $\int f = \int g$. (follows from 2).

4) If $a \leq f(x) \leq b$ then $amE \leq \int f \leq bmE$.

5) If A and B are disjoint sets of finite measure, then $\int_{A \cup B} f = \int_A f + \int_B f$.

4.5 Relation to Riemann Integral

Proposition 4.4. *If f is bounded with domain $[a, b]$ and is Riemann integrable, then it is Lebesgue integrable and the integrals are the same.*

Proposition 4.5. *A bouded function f with domain $[a, b]$ is Riemann integrable iff the set of points of discontinuities of f has measure zero.*

4.6 Convergence In Measure

We will develop here a notion that is more general than convegence a.e.

Definition 4.6. *A sequence f_n of measurable functions is said to converge to f in measure if, given $\epsilon > 0$, there is an N such that for all $n \geq N$ we have $m\{x : |f(x) - f_n(x)| \geq \epsilon\} < \epsilon$.*

Proposition 4.6. *Let f_n be a sequence of measurable functions that converge in measure to f. Then there is a subsequence f_n that converges to f almost everywhere.*

Corollary 4.3. *Let f_n be a sequence of measurable functions defined on a measurable set E of finite measure. Then f_n converges to f in measure iff every subsequence of f_n has in turn a subsequence that converges almost everywhere to f.*

Proposition 4.7. *Fatou's Lemma, the Monotone convergence theorem, and the Lebesgue convergence theorem remain valid if "convergence a.e." is replaced with "convergence in measure".*

5 Chapter 5: Differentiation and Integration

5.1 Definition Of Derivitive

Definition 5.1. *Let G be a collection of intervals. G covers a set E **in the sense of Vitali** if for each $\epsilon > 0$ and any $x \in E$, there is an interval $I \in G$ such that $x \in I$ and $l(I) < \epsilon$.*

Lemma 5.1. ***Vitali:** Let E be a set with finite outer measure and let G be a collection of intervals that cover E in the sense of Vitali. Then, given $\epsilon > 0$, there is a finite disjoint collection $\{I_1, ..., I_n\}$ of intervals in G such that $m^*[E \setminus \bigcup_n I_n] < \epsilon$.*

Definition 5.2. *Given a function f, the left/right upper/lower derivatives D^+f, D^-f, D_+f, D_-f are defined in the usual way. We say that f is **differentiable** at x if all four of these values are the same at x.*

Proposition 5.1. *If any one of D^+f, D^-f, D_+f, or D_-f is everywhere nonnegative on $[a, b]$ then f is nondecreasing on $[a, b]$.*

Theorem 5.1. *If f is a nondecreasing real-valued function on the interval $[a, b]$, then f is differentiable a.e. on $[a, b]$ and $\int_{[a,b]} f' \leq f(b) - f(a)$.*

5.2 Bounded Variation

Definition 5.3. *A function of **bounded variation** is defined in the usual way. The total (T), positive (P), and negative (N) variation are defiend in the usual way. (A function has bounded variation precicely when $T < \infty$, in which case $P < \infty$, $N < \infty$, and $T = P + N$).*

Theorem 5.2. *A function f is of bounded variation on $[a, b]$ iff it is the difference of two (strictly) monotone functions on $[a, b]$.*

Proposition 5.2. *If f is of bounded variation on $[a, b]$, then f' exists a.e. on $[a, b]$.*

5.3 Differentiation of an Integral

Lemma 5.2. *If f is integrable on $[a, b]$, then the function F defined by $F(x) = \int_a^x f(t)dt$ is a continuous function of bounded variation on $[a, b]$.*

Lemma 5.3. *If f is integrable on $[a, b]$ and $\int_a^x f(t)dt = 0$ for all $x \in [a, b]$, then $f(t) = 0$ a.e. in $[a, b]$.*

Lemma 5.4. *If f is bounded and measurable on $[a, b]$ and $F(x) = \int_a^x f(t)dt + F(a)$ then $F'(x) = f(x)$ for a.e. x in $[a, b]$.*

Theorem 5.3. *Let f be an integrable function on $[a, b]$, and suppose that $F(x) = F(a) + \int_a^x f(t)dt$. Then $F'(x) = f(x)$ a.e for x in $[a, b]$.*

5.4 Integral of a Derivative

Definition 5.4. *A function f with domain $[a,b]$ is said to be* ***absolutely continuous*** *if for all $\epsilon > 0$, there is a $\delta > 0$ such that for every sequence $\{(x_i x_i')\}_{i=1}^n$, $\sum_n |x_i - x_i'| < \delta$ implies $\sum_n |f(x_i) - f(x_i')| < \epsilon$.*

Lemma 5.5. *If f is absolutely continuous an $[a,b]$, then it is of bounded variation on $[a,b]$.*

Corollary 5.1. *If f is absolutely continuous then on $[a,b]$, then f has a derivative a.e. on $[a,b]$.*

Lemma 5.6. *If f is absolutely continuous on $[a,b]$ and $f'(x) = 0$ a.e. on $[a,b]$, then f is constant.*

Theorem 5.4. *A function F is an indefinite integral iff it is absolutely continuous.*

Corollary 5.2. *Every absolutely continuous function is the indefinite integral of its derivative.*

5.5 Convex Functions

Definition 5.5. *A function f with domain (a,b) is* ***convex*** *if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for all $x, y \in (a,b)$ and $\lambda \in [0,1]$.*

Lemma 5.7. *If f is convex with domain (a,b) and if $x, x', y, y' \in (a,b)$ are such that $x \leq x'$, $y \leq y'$, $x < y$, $x' < y'$, then $\frac{f(y)-f(x)}{y-x} \leq \frac{f(y')-f(x')}{y'-x'}$.*

Proposition 5.3. *Let f is convex with domain (a,b), then f is absolutely continuous on each closed subinterval of (a,b). The right and left derivatives of f exis at each point of (a,b) and are equal to each other.*

Proposition 5.4. *If f is a continuous function on (a,b), and if one derivative (say D^+) of f is nondecreasing, then f is convex.*

Corollary 5.3. *Let f have a second derivatice at each point of (a,b). Then f is convex on (a,b) iff $f''(x) \geq 0$ for each $a \in (a,b)$.*

Proposition 5.5. ***Jensen Inequality****: Let ϕ be a convex function on $(-\infty, \infty)$ and f an integrable function on $[0,1]$. Then $\int \phi(f(t))dt \geq \phi[\int f(t)dt]$.*

Corollary 5.4. *Let f be an integrable function on $[0,1]$. Then $\int exp(f(t))dt \geq exp[\int f(t)dt]$.*

6 Chapter 6: The Classical Banach Spaces

6.1 Personal Overview

Inner product space $\Rightarrow$ normed linear space $\Rightarrow$ metric space $\Rightarrow$ topological space. An inner product yields a norm, and a norm yields a metric. A norm that satisfies the parallegram law yields an inner product.

Definition 6.1. *A* ***Banach Space*** *is a complete normed linear space.*

Definition 6.2. *A* ***Hilbet Space*** *is a complete inner product space.*

Definition 6.3. *Let p be a positive real number. The space $L^p = L^p[0,1]$ is the set of measurable functions f with domain $[0,1]$ such that $\int |f|^p < \infty$ together with the natural addition and scalar multiplication operations.*

Definition 6.4. *Let f be a measurable function. The* ***essential supremum*** *is defined as follows: ess sup $f(t) = \inf\{M : m\{t : f(t) > M\} = 0\}$.*

Definition 6.5. *If f is a measurable function then $\| f \|_\infty$ is defined as ess sup $f(t)$. The $L^\infty[0,1]$ space is defined to be the set of measurable functions with domain $[0,1]$ such that $\| f \|_\infty < \infty$.*

6.2 The Minkowski and Hölder Inequalities

Lemma 6.1. *If p is a positive real and $a, b \in \mathbb{R}$, then $|a+b|^p \leq 2^p(|a|^p + |b|^p)$.*

The above lemma demonstates that the L^p spaces are indeed closed under addition.

Theorem 6.1. ***Minkowski Inequality**: Let $p \in (0, \infty]$ and $f, g \in L^p$. If $1 \leq p \leq \infty$, then $\| f + g \|_p \leq \| f \|_p + \| g \|_p$ (with equality only if one function is a multiple of the other). If $0 < p < 1$, then $\| f + g \|_p \geq \| f \|_p + \| g \|_p$.*

The Minkowski inequality is thus showing that the norms of the L^p spaces ($p \geq 1$) indeed satisfy the triangle inequality.

Theorem 6.2. ***Hölder Inequality**: If p and q are positive real numbers (or ∞) such that $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in L^p$ and $g \in L^q$, then $f \cdot g \in L^1$ and $\int |fg| \leq \| f \|_p \| g \|_q$. Equality holds iff for some constants a, b, not both zero, we have $a|f|^p = b|g|^q$ a.e.*

Note: The Hölder Inequality for $p, q = 2$ is called the cauchy schwarz inequality, which implies the triangle inequality but we do not need this because we already proved the Minkowski Inequality.

6.3 Convergence and Completeness

Proposition 6.1. *A normed linear space X is complete iff every absolutely summable series is summable.*

Theorem 6.3. ***Riesz-Fischer**: The $L^p[0,1]$ spaces are complete.*

6.4 Approximation in L^p

Proposition 6.2. *Given $f \in L^p$, $1 \leq p < \infty$, and $\epsilon > 0$, there is a bounded measurable function f_M with $|f_M| \leq M$ and $\| f - f_M \|_p < \epsilon$.*

It appears as though the condition $p < \infty$ above can be strengthened to $p \leq \infty$ because of the following proposition:

Proposition 6.3. *Given $f \in L^p$, $1 < p \leq \infty$ and $\epsilon > 0$, there is a step function ϕ and a continuous function ψ such that $\| f - \phi \|_p < \epsilon$ and $\| f - \psi \|_p < \epsilon$.*

Definition 6.6. *Let $\Delta = \{z_0, ..., z_m\}$ be a subdivision of $[a, b]$ and f be an integrable function on $[a, b]$. The function ϕ_Δ defined by $\phi_\Delta = \frac{1}{z_{k+1} - z_k} \int_{z_k}^{z_k + 1} f(t)dt$ is called the Δ-appriximant to f in mean.*

Proposition 6.4. *Let $f \in L^p$. Then the Δ-approximant ϕ_Δ to f converges to f in L^p as the length δ of the longest subinterval in Δ approaches zero.*

Corollary 6.1. *The Δ-approximant ϕ_Δ converges to f in measure as $\delta \to 0$.*

6.5 Bounded Linear Functionals on the L^p spaces

Definition 6.7. *If F maps from L^p to $\mathbb{R}$ then $\| F \| =_{df} \sup \frac{|F(f)|}{\|f\|}$.*

Proposition 6.5. *Each function g in L^q defines a bounded linear functional F on L^p by $F(f) = \int fg$. We have $\| F \| = \| g \|_q$.*

Lemma 6.2. *Let g be an integrable function on $[0, 1]$, and suppose that there is a constant M such that $|\int fg| \leq M \| f \|_p$ for all bounded measurable functions f. Then g is in L^q, and $\| g \|_q \leq M$.*

Theorem 6.4. ***Riesz Representation Theorem**: Let F be a bounded linear functional on L^p, $1 \leq p < \infty$. Then there is a function g on L^q such that $F(f) = \int fg$. We also have $\| F \| = \| g \|_q$.*