The Cohomology Cup Product

Dan Hathaway

April 24, 2011

1 The Cup Product

1.1 Defining The Cup Product and Showing it is Well Defined

Consider cohomology with coefficients in some fixed ring R. This is different from what we have been studying so far in class, because up till now our coefficients were in an arbitrary abelian group. We can take the direct sum $\bigoplus_{n\geq 0} H^n(X)$ of all cohomology groups of a topological space X to obtain an abelian group. What we would like to do is add a multiplication operation on this group to get a ring. This multiplication will be the *cup product*.

Given $\psi \in Z^n(X)$, let $[\psi]$ represent the cohomology class of ψ. Recall that $C^n(X) = \text{Hom}(C_n(X), R)$, $\phi \in Z^n(X)$ means $\delta(\phi) = 0$, and $\phi \in B^n(X)$ means $\phi = \delta(\phi')$ for some $(n-1)$-cochain $\phi' \in C^{n-1}(X)$. To define the cup product $\smile$ on $\bigoplus_{n\geq 0} H^n(X)$, it suffices to define $[\phi] \smile [\psi]$ for $[\phi] \in H^n(X)$ and $[\psi] \in H^m(X)$ where $n, m \geq 0$ are arbitrary (because then $\smile$ can be defined component wise). To do this, we will use the symbol $\smile$ to denote a product $\phi \smile \psi$ of n and m cohains, and we will show that this operation is well defined on cohomology classes. Given $\phi \in C^n(X)$ and $\psi \in C^m(X)$, define $\phi \smile \psi$ to be the $(n+m)$-cochain that satisfies the following formula for all $\sigma : \Delta^{n+m} \to X$:

$$(\phi \smile \psi)(\sigma) = \phi(\sigma|[v_0, ..., v_n])\psi(\sigma|[v_n, ..., n_{n+m}]).$$

This formula makes sense because $\phi(\sigma|[v_0, ..., v_n])$ and $\psi(\sigma|[v_n, ..., v_{n+m}])$ are both elements of R, so they can be multiplied.

What we want is an induced $\smile$ operation

$$H^n(X) \times H^m(X) \to H^{n+m}(X).$$

To get this, we must show that the $\smile$ operation

$$C^n(X) \times C^m(X) \to C^{n+m}(X)$$

maps $Z^n(X) \times Z^m(X)$ to Z^{n+m}, and if $[\phi] = [\phi']$ and $[\psi] = [\psi']$, then $[\phi \smile \psi] = [\phi' \smile \psi']$. Both of these facts are implied by the following Lemma:

Lemma: If $\phi \in C^n(X)$ and $\psi \in C^m(X)$, then $\delta(\phi \smile \psi) = \delta\phi \smile \psi + (-1)^n \phi \smile \delta\psi$.
Proof: We will compute $\delta(\phi \smile \psi)(\sigma)$, $(\delta\phi \smile \psi)(\sigma)$, and $(\phi \smile \delta\psi)(\sigma)$ separately for an arbitrary $\sigma : \Delta^{n+m+1} \to X$. Computing $\delta(\phi \smile \psi)$:

$$\delta(\phi \smile \psi)(\sigma) = (\phi \smile \psi)(\partial\sigma) = (\phi \smile \psi)(\sum_{i=0}^{n+m+1} (-1)^i \sigma|[v_0, ..., \hat{v}_i, ..., v_{n+m+1}]) =$$

$$\sum_{i=0}^{n+m+1} (-1)^i (\phi \smile \psi)(\sigma|[v_0, ..., \hat{v}_i, ..., v_{n+m+1}]).$$

The last equality follows because $\phi \smile \psi$ is a homomorphism. Computing $(\delta\phi \smile \psi)(\sigma)$:

$$(\delta\phi \smile \psi)(\sigma) = (\delta\phi)(\sigma|[v_0, ..., v_{n+1}])\psi(\sigma|[v_{n+1}, ..., v_{n+m+1}]) = \phi(\partial(\sigma|[v_0, ..., v_{n+1}]))\psi(...) =$$

$$\phi(\sum_{i=0}^{n+1}(-1)^i\sigma|[v_0,...,\hat{v}_i,...,v_{n+1}])\psi(...) = \sum_{i=0}^{n+1}(-1)^i\phi(\sigma|[v_0,...,\hat{v}_i,...,v_{n+1}])\psi(...).$$

Computing $(\phi \smile \delta\psi)(\sigma)$ in the same way:

$$(\phi \smile \delta\psi)(\sigma) = \phi(\sigma|[v_0,...,v_n])(\delta\psi)(\sigma|[v_n,...,v_{n+m+1}]) = \phi(...)\psi(\partial(\sigma|[v_n,...,v_{n+m+1}])) =$$

$$\phi(...)\psi(\sum_{i=n}^{n+m+1}(-1)^{i-n}\sigma|[v_n,...,\hat{v}_i,...,v_{n+m+1}]) = \sum_{i=n}^{n+m+1}(-1)^{i-n}\phi(...)\psi(\sigma|[v_n,...,\hat{v}_i,...,v_{n+m+1}]).$$

Thus,

$$(\delta\phi \smile \psi)(\sigma) + (-1)^n(\phi \smile \delta\psi)(\sigma) = \sum_{i=0}^{n+1}(-1)^i\phi(\sigma|[v_0,...,\hat{v}_i,...,v_{n+1}])\psi(\sigma|[v_{n+1},...,v_{n+m+1}])+$$

$$(-1)^n\sum_{i=n}^{n+m+1}(-1)^{i-n}\phi(\sigma|[v_0,...,v_n])\psi(\sigma|[v_n,...,\hat{v}_i,...,v_{n+m+1}]).$$

Notice how the last term of the first sum cancels with the first term of the second sum. Comparing the resulting sum to what we calculated $\delta(\phi \smile \psi)$ to be, we find that $(\delta\phi \smile \psi)(\sigma) + (-1)^n(\phi \smile \delta\psi)(\sigma) = \delta(\phi \smile \psi)$. This completes the proof of the lemma. □

With this lemma in place, it is clear that if $\phi \in Z^n(X)$ and $\psi \in Z^m(X)$, then $\phi \smile \psi \in Z^{n+m}(X)$: if $\delta\phi = 0$ and $\delta\psi = 0$, then $\delta(\phi \smile \psi) = \delta\phi \smile \psi + (-1)^n\phi \smile \delta\psi = 0 \smile \psi + (-1)^n\phi \smile 0 = 0 + 0 = 0$. Thus, $\smile$ restricts to a map from $Z^n(X) \times Z^m(X)$ to $Z^{n+m}(X)$.

Suppose now that $\phi, \phi' \in Z^n(X)$, $\psi, \psi' \in Z^m(X)$, $\phi - \phi' \in B^n(X)$, and $\psi - \psi' \in B^m(X)$. We want to show that $[\phi \smile \psi] = [\phi' \smile \psi']$. Let $\phi - \phi' = \delta\tilde{\phi}$ for some $\tilde{\phi} \in C^{n+1}(X)$. The lemma gives us $\delta(\tilde{\phi} \smile \psi) = \delta\tilde{\phi} \smile \psi + (-1)^n\phi \smile \delta\psi$. Since $\psi \in Z^n(X)$, this means that $\delta\psi = 0$ so $\delta(\tilde{\phi} \smile \psi) = \delta\tilde{\phi} \smile \psi$. Thus, $\delta\tilde{\phi} \smile \psi$ is a coboundary. That is, $\delta\tilde{\phi} \smile \psi = (\phi - \phi') \smile \psi = (\phi \smile \psi) - (\phi' \smile \psi)$ is a coboundary. That is, $[\phi \smile \psi] = [\phi' \smile \psi]$. In the same way, we can show $[\phi' \smile \psi] = [\phi' \smile \psi']$. This establishes $[\phi \smile \psi] = [\phi' \smile \psi']$. That is, $\smile$ is a well defined map from $H^n(X) \times H^m(X)$ to $H^{n+m}(X)$.

1.2 Cohomology Ring

With the cup product defined, it is worth checking that this operation on $\bigoplus_{n\geq 0} H^n(X)$ indeed gives us a ring structure. We have that $\smile$ is associative because given $[\phi] \in H^n(X)$, $[\psi] \in H^m(X)$, and $[\gamma] \in H^k(X)$, we have

$$((\phi \smile \psi) \smile \gamma)(\sigma) = \phi(\sigma|[v_0,...,v_n])\psi(\sigma|[v_n,...,v_{n+m}])\gamma(\sigma|[v_{n+m},...,v_{n+m+k}]) = (\phi \smile (\psi \smile \gamma))(\sigma)$$

so $([\phi] \smile [\psi]) \smile [\gamma] = [\phi] \smile ([\psi] \smile [\gamma])$. We also have that $\smile$ distributes with addition, because if $\phi, \phi' \in Z^n(X)$ and $\psi \in Z^m(X)$, then

$$((\phi+\phi') \smile \psi)(\sigma) = (\phi+\phi')(\sigma|[v_0,...,v_n])\psi(\sigma|[v_n,...,v_{n+m}]) = \phi(...)\psi(...)+\phi'(...)\psi(...) = (\phi \smile \psi+\phi' \smile \psi)(\sigma).$$

The same is true for the second argument. Thus, we do indeed have a ring structure on cohomology. In fact, we have more than just a ring structure: $\bigoplus_{n\geq 0} H^n(X)$ forms an R-algebra (which can be easily verified).

1.3 Taking the Cohomology Ring is Functorial

We have shown that every topological space has associated with it a cohomology ring. The question arises as to if there is a ring homomorphism induced by a continuous function between topological spaces. This turns out to be the case.

That is, let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. For each n, we have an induced group homomorphism $f_n^* : H^n(Y) \to H^n(X)$. This induces a group homomorphism $f^* : \bigoplus_{n\geq 0} H^n(Y) \to \bigoplus_{n\geq 0} H^n(X)$. We claim that this map f^* is also a ring homomorphism. This is because if ϕ is an i-cochain on Y and ψ is a j-cochain on Y, then

$$(f^*(\phi) \smile f^*(\psi))(\sigma) = (f^*(\phi))(\sigma|[v_0, ..., v_i])(f^*(\psi))(\sigma|[v_i, ..., v_{i+j}]) =$$

$$\phi((f \circ \sigma)|[v_0, ..., v_i])\psi((f \circ \sigma)|[v_i, ..., v_{i+j}]) = (\phi \smile \psi)(f \circ \sigma) = f^*(\phi \smile \psi)(\sigma).$$

Thus, on cohomology classes we have $[f^*(\phi)] \smile [f^*(\psi)] = [f^*(\phi \smile \psi)]$. Thus, for elements $a, b \in \bigoplus_{n\geq 0} H^n(Y)$ we have $f^*(a) \smile f^*(b) = f^*(a \smile b)$. That is, f^* is a ring homomorphism.

We have that if X, Y, and Z are topological spaces with the continuous maps $f : X \to Y$ and $g : Y \to Z$, then $f_n^* \circ g_n^* = (f \circ g)_n^*$, for all n. This means that $f^* \circ g^* = (f \circ g)^*$. The fact that these maps commute is not affected by the additional ring structure that is put on $\bigoplus_{n\geq 0} H^n(X)$, $\bigoplus_{n\geq 0} H^n(Y)$, and $\bigoplus_{n\geq 0} H^n(Z)$. Thus, the operation F of taking the cohomology ring is a (contravariant) functor from the category of topological spaces to the category of rings.

1.4 Cup Product Is Supercommutative

Consider the cohomology ring $\bigoplus_{n\geq 0} H^n(X)$. Let $a \in H^i$ and $b \in H^j$. As long as R is commutative (a fact which we are assuming), it turns out that $a \smile b = (-1)^{ij} b \smile a$. We mention this fact here because we will need this for having a well defined multiplication operation on the tensor product of two cohomology rings. We omit the proof of supercommutivity for brevity.

2 Computing the Cohomology Ring of Simplicial Complexes

The definition we gave for $\smile$ can be modified to apply to simplicial complexes. Using this definition, we will compute the cohomology ring of various simplicial complexes.

2.1 Using $\mathbb{Z}$ as a Coefficient Ring

From now on in this section, we will use $\mathbb{Z}$ instead of an arbitrary ring R. Recall an important tool from a past homework assignment for calculating the cohomology of a simplicial complex given that we are using $\mathbb{Z}$ for our coefficient group: if all the homology groups $H_n(X)$ are free abelian, then the canonical homomorphism from the abelian group $H^n(X)$ to the abelian group $\text{Hom}(H_n(X), \mathbb{Z})$ is an isomorphism. We will use this result to compute the cohomology of a space as a first step before we investigate the ring structure induced by the cup product.

2.2 Cohomology Ring of a Point

First things first, we will compute the cohomology ring of X where X is a space with a single point. Once this is done, we will know the cohomology ring of any contractible space. In class we showed $H_0(X) = \mathbb{Z}$ and $H_i(X) = 0$ for $i \neq 0$. Since every homology group of X is free abelian, we have $H^i(X) \cong \text{Hom}(H_i(X), \mathbb{Z})$ for all i. Since $H_0(X) = \mathbb{Z}$, we have $H^0(X) \cong \text{Hom}(\mathbb{Z}, \mathbb{Z})$. Since $H_i(X) = 0$ for $i \geq 0$, we have $H^i(X) \cong \text{Hom}(0, \mathbb{Z}) = 0$. Since the cup product maps $H^n(X) \times H^m(X)$ to $H^{n+m}(X)$ and $H^{n+m}(X) = 0$ for $n+m \geq 1$, the only interesting case is when $n+m = 0$. That is, when $n = m = 0$. To see what $\smile$ does to elements of $H^0(X) \times H^0(X)$, let ϕ and ψ be two arbitrary 0-cocycles representing elements of $H^0(X)$. That is, $\phi, \psi \in \text{Hom}(C_0(X), \mathbb{Z})$ and $\delta\phi = \delta\psi = 0$. Thus, ϕ and ψ are both constant functions. Their cup product $\phi \smile \psi$ is just the constant function whose value is the product of their values. This completes the description of the cohomology ring structure of X.

2.3 Cohomology Ring of A Disjoint Union of Spaces

Suppose the simplicial complex X is the disjoint union of two simplicial complexes X_1 and X_2. Suppose that we have computed the cohomology ring of X_1 and X_2. We claim that we know the cohomology ring of X. To see why, suppose $\phi \in C^n(X_1)$ and $\psi \in C^m(X_2)$. ϕ can be viewed as an element of $C^n(X)$ that maps every n-simplex of X_2 to 0. Similarly, ψ can be viewed as an element of $C^m(X)$ that maps every m-simplex of X_1 to 0. We have that for every $(n+m)$-simplex σ in X, $(\phi \smile \psi)(\sigma) = \phi(\sigma|[v_0, ..., v_n])\psi(\sigma|[v_n, ..., v_{n+m}])$. Since every $(n+m)$-simplex σ is either in X_1 or X_2, we have that either $\psi(\sigma|[v_n, ..., v_{n+m}]) = 0$ or $\phi(\sigma|[v_0, ..., v_n]) = 0$. Thus, $(\phi \smile \psi)(\sigma) = 0$ for all σ. Thus, $\phi \smile \psi = 0$. Because of this, the cohomology ring of X is the direct sum of the cohomology ring of X_1 and the cohomology ring of X_2.

2.4 Cohomology Ring of S^n

Consider a triangulation X of S^n. That is, X is a simplicial complex and $|X|$ is homeomorphic to S^n. In class, we showed that $H_0(X) = \mathbb{Z}$, $H_n(X) = \mathbb{Z}$, and $H_i(X) = 0$ for $i \neq 0, n$. Since the homology groups of X are all free abelian, we have $H^0(X) \cong \text{Hom}(H_0(X), \mathbb{Z}) \cong \mathbb{Z}$, $H^n(X) \cong \text{Hom}(H_n(X), \mathbb{Z}) \cong \mathbb{Z}$, and $H^i(X) \cong \text{Hom}(H_i(X), \mathbb{Z}) = 0$ for $i \neq 0, n$. Thus, the cup product is only interesting in the $H^0(X) \times H^0(X) \to H^0(X)$, $H^0(X) \times H^n(X) \to H^n(X)$, and $H^n(X) \times H^0(X) \to H^n(X)$ cases. The first case is identical to the example of the space with only one point. It can be seen that all path connected spaces have the same cup product structure on $H^0(X) \times H^0(X)$. To understand the cup product on $H^0(X) \times H^n(X) \to H^n(X)$, suppose $\phi \in \text{Hom}(C_0(X), \mathbb{Z})$ and $\psi \in \text{Hom}(C_n(X), \mathbb{Z})$ are arbitrary cocycles ($\delta\phi = 0, \delta\psi = 0$) representing elements of $H^0(X)$ and $H^n(X)$ respectively. We have that ϕ is a constant function. Thus, $\phi \smile \psi$ is just ψ multiplied by the value that is the constant output value of ϕ. The same can be said about $H^n(X) \to H^0(X) \to H^n$. This completes the description of the Cohomology ring structure of S^n.

2.5 Cohomology Ring of $S^1 \times S^1$

We will now consider a less trivial example. Let T be the triangulation of $S^1 \times S^1$ given in the hand drawn picture. Consider the 0-cochain g, the two 1-cochains f^1 and f^2, and the 2-cochain h. We claim that $\delta g = 0$, $\delta f^1 = 0$, and $\delta f^2 = 0$ (we automatically have $\delta h = 0$ because there are no 3-simplices). To see that $\delta g = 0$, note that $(\delta g)([v_i, v_j]) = g([v_j]) - g([v_i]) = 1 - 1 = 0$ for any 1-simplex $[v_i, v_j]$ of T (where $i < j$).

To show that $\delta f^1 = 0$, we will show $(\delta f^1)(\sigma) = 0$ for every 2-simplex σ of T. The only 2-simplices we need to check are those that have at least one face in the following set of 1-simplices:

$$\{[v_2, v_3], [v_2, v_6], [v_5, v_6], [v_5, v_9], [v_8, v_9], [v_3, v_8]\}.$$

These 1-simplices are the edges with a squiggly line through them in the drawn triangulation. Performing the verification that $\delta f^1 = 0$:

$$\begin{aligned}
(\delta f^1)([v_2, v_3, v_6]) &= f^1([v_2, v_3]) + f^1([v_3, v_6]) - f^1([v_2, v_6]) = 1 + 0 - 1 = 0 \\
(\delta f^1)([v_2, v_5, v_6]) &= f^1([v_2, v_5]) + f^1([v_5, v_6]) - f^1([v_2, v_6]) = 0 + 1 - 1 = 0 \\
(\delta f^1)([v_5, v_6, v_9]) &= f^1([v_5, v_6]) + f^1([v_6, v_9]) - f^1([v_5, v_9]) = 1 + 0 - 1 = 0 \\
(\delta f^1)([v_5, v_8, v_9]) &= f^1([v_5, v_8]) + f^1([v_8, v_9]) - f^1([v_5, v_9]) = 0 + 1 - 1 = 0 \\
(\delta f^1)([v_3, v_8, v_9]) &= f^1([v_3, v_8]) + f^1([v_8, v_9]) - f^1([v_3, v_9]) = (-1) + 1 - 0 = 0 \\
(\delta f^1)([v_2, v_3, v_8]) &= f^1([v_2, v_3]) + f^1([v_3, v_8]) - f^1([v_2, v_8]) = 1 + (-1) - 0 = 0.
\end{aligned}$$

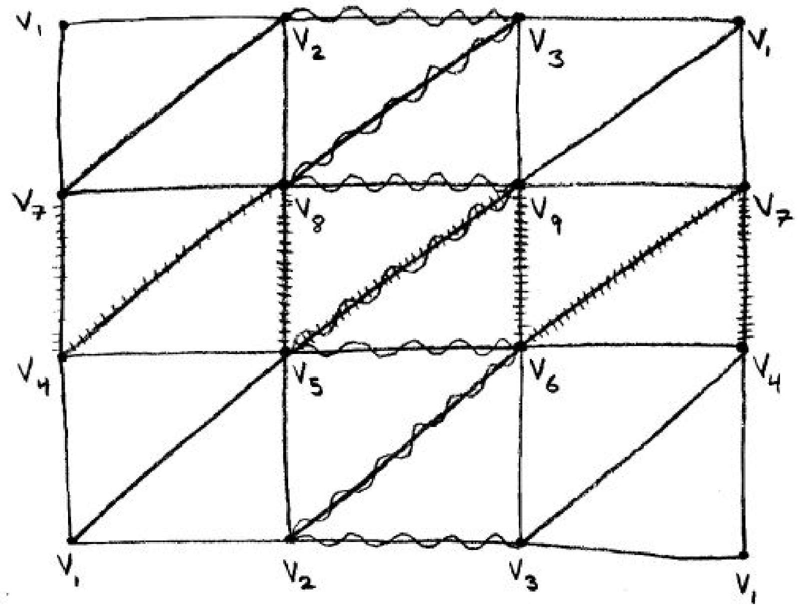
To show that $\delta f^2 = 0$, we need only show that $(\delta f^2)(\sigma) = 0$ for all 2-simplices σ that have at least one face in the following set of simplices (the edges with dashed lines in the drawing):

$$\{[v_4, v_7], [v_4, v_8], [v_5, v_8], [v_5, v_9], [v_6, v_9], [v_6, v_7]\}.$$

Performing this verification:

$$\begin{aligned}
(\delta f^2)([v_4, v_7, v_8]) &= f^1([v_4, v_7]) + f^1([v_7, v_8]) - f^1([v_4, v_8]) = 1 + 0 - 1 = 0 \\
(\delta f^2)([v_4, v_5, v_8]) &= f^2([v_4, v_5]) + f^2([v_5, v_8]) - f^2([v_4, v_8]) = 0 + 1 - 1 = 0 \\
(\delta f^2)([v_5, v_8, v_9]) &= f^2([v_5, v_8]) + f^2([v_8, v_9]) - f^2([v_5, v_9]) = 1 + 0 - 1 = 0 \\
(\delta f^2)([v_5, v_6, v_9]) &= f^2([v_5, v_6]) + f^2([v_6, v_9]) - f^2([v_5, v_9]) = 0 + 1 - 1 = 0 \\
(\delta f^2)([v_6, v_7, v_9]) &= f^2([v_6, v_7]) + f^2([v_7, v_9]) - f^2([v_6, v_9]) = 1 + 0 - 1 = 0 \\
(\delta f^2)([v_4, v_6, v_7]) &= f^2([v_4, v_6]) + f^2([v_6, v_7]) - f^2([v_4, v_7]) = 0 + 1 - 1 = 0.
\end{aligned}$$

TRIANGULATION T OF $S^1 \times S^1$:



$f^1([v_2, v_3]) = 1$
$f^1([v_2, v_6]) = 1$
$f^1([v_5, v_6]) = 1$
$f^1([v_5, v_9]) = 1$
$f^1([v_8, v_9]) = 1$
$f^1([v_3, v_8]) = -1$
$f^1(\text{ANY OTHER 1-SIMPLEX}) = 0$

$f^2([v_4, v_7]) = 1$
$f^2([v_4, v_8]) = 1$
$f^2([v_5, v_8]) = 1$
$f^2([v_5, v_9]) = 1$
$f^2([v_6, v_9]) = 1$
$f^2([v_6, v_7]) = 1$
$f^2(\text{ANY OTHER 1-SIMPLEX}) = 0$

$h([v_5, v_6, v_9]) = 1$
$h(\text{ANY OTHER 2-SIMPLEX}) = 0$

$g(\text{ANY 0-SIMPLEX}) = 1$

CLAIM: $\delta g = 0$, $\delta f^1 = 0$, $\delta f^2 = 0$
$[g]$ GENERATES $H^0(T)$,
$[f^1]$ AND $[f^2]$ GENERATE $H^1(T)$, AND
$[h]$ GENERATES $H^2(T)$.

We have now shown that g, f^1, f^2, and h are all cocycles. We will show that their cohomology classes generate the cohomology groups of T. First, we will give a direct argument that $[h]$ generates $H^2(T)$ (as an example of how to make such a claim). Let h' be an arbitrary 2-cochain:

$$h' = \sum_i n_i \chi_{\sigma_i}$$

where each $n_i \in \mathbb{Z}$ and χ_{σ_i} is the function where $\chi_{\sigma_i}(\sigma_i) = 1$ but $\chi_{\sigma_i}(\sigma) = 0$ for any other $\sigma \neq \sigma_i$. We claim that h' is cohomologous to some cocycle of the form $(\sum_i \tilde{n}_i)h$ where each $\tilde{n}_i$ is either n_i or $-n_i$. To verify this, we need only show that given two adjacent 2-simplices σ_i and σ_j, we have χ_{σ_i} is cohomologous to either χ_{σ_j} or $-\chi_{\sigma_j}$. Once we have this, the result follows immediately by indunction. Let $\sigma_1 = [v_5, v_8, v_9]$ and $\sigma_2 = [v_5, v_6, v_9]$. We will show that χ_{σ_1} is cohomologous to $-\chi_{\sigma_2}$, and this argument generalizes to any two adjacent 2-simplices of T. Consider the 1-cochain k defined by $k([v_5, v_9]) = 1$ but $k(\tau) = 0$ for any other 1-simplex $\tau \neq [v_5, v_9]$. We have that δk is a 2-cochain such that

$$(\delta k)(\sigma_1) = (\delta k)([v_5, v_8, v_9]) = k([v_5, v_8]) + k([v_8, v_9]) - k([v_5, v_9]) = 0 + 0 - 1 = -1$$
$$(\delta k)(\sigma_2) = (\delta k)([v_5, v_6, v_9]) = k([v_5, v_6]) + k([v_6, v_9]) - k([v_5, v_9]) = 0 + 0 - 1 = -1$$

and $(\delta k)(\sigma) = 0$ for any other 2-simplex $\sigma \neq \sigma_1, \sigma_2$. Thus, we have shown that $\delta k = -\chi_{\sigma_2} - \chi_{\sigma_1}$. That is, $\delta k = (-\chi_{\sigma_2}) - (\chi_{\sigma_1})$, so $-\chi_{\sigma_2}$ is cohomologous to χ_{σ_1}. Since this argument generalizes to any other two adjacent 2-simplices σ_i and σ_j, it follows that h' is cohomologous to some integer multiple of h. Thus, $[h]$ generates $H^2(T)$.

It is immediate that $[g]$ generates $H^0(T)$ because every element g' of $Z^0(T)$ is a constant function on the 0-simplices, so g' is a $\mathbb{Z}$-multiple of g.

We will now show that $[f^1]$ and $[f^2]$ generate $H^1(T)$. In a homework assignment, we showed $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}^2$, and $H_2(X) = \mathbb{Z}$. Since these homology groups are all free abelian, we have that the canonical homomorphisms $\psi_i : H^i(T) \to \mathrm{Hom}(H_i(T), \mathbb{Z})$ are isomorphisms for $i = 0, 1, 2$. In particular, the map $\psi_1 : H^1(T) \to \mathrm{Hom}(H_1(T), \mathbb{Z})$ is an isomorphism. If we can show that $\psi_1([f^1])$ and $\psi_1([f^2])$ generate $\mathrm{Hom}(H_1(T), \mathbb{Z})$, we will have that $[f^1]$ and $[f^2]$ generate $H^1(T)$. Consider the homomorphism $\psi_1([f^1]) \in \mathrm{Hom}(H_1(T), \mathbb{Z})$. In a previous homework assignment, we showed that

$$a = [v_1, v_2] + [v_2, v_3] - [v_1, v_3]$$

and

$$b = [v_1, v_4] + [v_4, v_7] - [v_1, v_7]$$

are such that $[a]$ and $[b]$ generate $H_1(T)$. To determine the value of the function $\psi_1([f^1])$ on $[a]$, we simply plug a into the function $f^1 : C_1(T) \to \mathbb{Z}$. We can do this for both $\psi_1([f^1])$ and $\psi_1([f^2])$ with both $[a]$ and $[b]$ as arguments:

$$\psi_1([f^1])([a]) = f^1(a) = f^1([v_1, v_2]) + f^1([v_2, v_3]) - f^1([v_1, v_3]) = 0 + 1 - 0 = 1$$
$$\psi_1([f^1])([b]) = f^1(b) = f^1([v_1, v_4]) + f^1([v_4, v_7]) - f^1([v_1, v_7]) = 0 + 0 - 0 = 0$$
$$\psi_1([f^2])([a]) = f^2(a) = f^2([v_1, v_2]) + f^2([v_2, v_3]) - f^2([v_1, v_3]) = 0 + 0 - 0 = 0$$
$$\psi_1([f^2])([b]) = f^2(b) = f^2([v_1, v_4]) + f^2([v_4, v_7]) - f^2([v_1, v_7]) = 0 + 1 - 0 = 1.$$

From these calculations (and the fact that $[a]$ and $[b]$ generate $H_1(T)$), it is clear that $\psi_1([f^1])$ and $\psi_1([f^2])$ generate $\mathrm{Hom}(H_1(T), \mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z}) \cong \mathbb{Z}^2$. Since ψ_1 is an isomorphism, we have that $[f^1]$ and $[f^2]$ generate $H^1(T)$.

We have now finished showing that $[g]$ generates $H^0(T)$, $[f^1]$ and $[f^2]$ generate $H^1(T)$, and $[h]$ generates $H^2(T)$. We will now compute the cup product structure of T using these generators. First, we have the obvious facts whose proof we have discussed before:

$$[g] \smile [g] = [g]$$
$$[g] \smile [f^1] = [f^1] \smile [g] = [f^1]$$
$$[g] \smile [f^2] = [f^2] \smile [g] = [f^2]$$
$$[g] \smile [h] = [h] \smile [g] = [h]$$
$$[h] \smile [f^1] = [f^1] \smile [h] = 0$$
$$[h] \smile [f^2] = [f^2] \smile [h] = 0$$
$$[h] \smile [h] = 0.$$

What is more interesting are products $[f^1] \smile [f^2]$ and $[f^2] \smile [f^1]$. For the 2-simplex $[v_5, v_6, v_9]$, we have

$$(f^1 \smile f^2)([v_5, v_6, v_9]) = f^1([v_5, v_6]) f^2([v_6, v_9]) = (1)(1) = 1.$$

For any other 2-simplex, σ, it can be seen that $(f^1 \smile f^2)(\sigma) = 0$. Thus, we have $f^1 \smile f^2 = \chi_{[v_5, v_6, v_9]} = h$. That is, $[f^1] \smile [f^2] = [h]$. On the other hand, to compute $f^2 \smile f^1$, notice that

$$(f^2 \smile f^1)([v_5, v_8, v_9]) = f^2([v_5, v_8]) f^1([v_8, v_9]) = (1)(1) = 1.$$

On the other hand, $(f^2 \smile f^1)([\sigma]) = 0$ for any other 2-simplex σ. Thus, we have $f^2 \smile f^1 = \chi_{[v_5,v_8,v_9]}$. We showed previously that $\chi_{[v_5,v_8,v_9]}$ is cohomologous to $-\chi_{[v_5,v_6,v_9]} = -h$. Thus, $[f^2] \smile [f^1] = -[h]$. That is, we have shown the following:

$$[f^1] \smile [f^2] = [h]$$
$$[f^2] \smile [f^1] = -[h].$$

This agrees with our previous claim that the cup product is supercommutative. This completes our calculation of the cohomology ring structure of the triangulation T of $S^1 \times S^1$. Later, we will show that what we have found here agrees with a formula for computing the cohomology ring of the product of two spaces.

3 Cross Product (External Cup Product)

3.1 Defining the Cross Product

Let X and Y be topological spaces. We have the (continuous) projection maps $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ defined by $p_1(x, y) = x$ and $p_2(x, y) = y$. These induce ring homomorphisms: $p_1^* : \bigoplus_{n\geq 0} H^n(X) \to \bigoplus_{n\geq 0} H^n(X \times Y)$ and $p_2^* : \bigoplus_{n\geq 0} H^n(Y) \to \bigoplus_{n\geq 0} H^n(X \times Y)$ (remember that taking the cohomology ring is a contravariant functor). We wish to combine these homomorphisms together somehow.

Given an i-cocycle ϕ on X and a j-cocycle ψ on Y, we can certainly define the map

$$H^i(X) \times H^j(Y) \xrightarrow{\times} H^{i+j}(X \times Y)$$

by $\phi \times \psi = p_1^*(\phi) \smile p_2^*(\psi)$ (because $p_1^*(\phi)$ is an i-cocycle on $X \times Y$ and $p_2^*(\psi)$ is a j-cocycle on $X \times Y$). Since p_1^* and p_2^* are both R-module homomorphisms, this map is R-bilinear. From this map we can define the *cross product*, otherwise known as the *external cup product*, termwise:

$$\left(\bigoplus_{n\geq 0} H^n(X)\right) \times \left(\bigoplus_{n\geq 0} H^n(Y)\right) \longrightarrow \bigoplus_{n\geq 0} H^n(X \times Y).$$

This map is R-bilinear.

3.2 Defining a Ring Homomorphism out of the Tensor Product

For ease of notation, let $\tilde{X} = \bigoplus_{n\geq 0} H^n(X)$ and $\tilde{Y} = \bigoplus_{n\geq 0} H^n(Y)$. If we wanted, we could define a componentwise multiplicative structure on $\tilde{X} \times \tilde{Y}$, but this would not be useful because the cross product would not be a ring homomorphism out of this space. Before jumping the gun and talking about ring homomorphisms, we should note that other than in trivial cases, the cross product map will not even be an R-module homomorphism. To remedy the situation, we use the fact that an R-bilinear map out of the product of R-modules induces an R-module homomorphism out of the tensor product of those spaces:

$$\begin{array}{ccc} \tilde{X} \times \tilde{Y} & \longrightarrow & \bigoplus_{n\geq 0} H^n(X \times Y) \\ \downarrow & \nearrow_{f} & \\ \tilde{X} \otimes \tilde{Y} & & \end{array}$$

Miraculously, we can define a multiplication operation on the tensor product by the following definition for simple tensors:

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd.$$

We will show that this multiplication is well defined and that this causes f to be a ring homomorphism.

To show that this multiplication operation m'' is well defined, what we want is a well defined map m'' that causes the following diagram to commute (q' and q'' are the canonical maps):

$$\begin{array}{ccccc} (\tilde{X} \times \tilde{Y}) \times (\tilde{X} \times \tilde{Y}) & \xrightarrow{q'} & (\tilde{X} \otimes \tilde{Y}) \times (\tilde{X} \times \tilde{Y}) & \xrightarrow{q''} & (\tilde{X} \otimes \tilde{Y}) \times (\tilde{X} \otimes \tilde{Y}) \\ \downarrow m & \swarrow_{m'} & & \swarrow_{m''} & \\ \tilde{X} \otimes \tilde{Y}. & & & & \end{array}$$

The map m is defined by $m((a,b),(c,d)) = (-1)^{|b||c|}ac \otimes bd$ for $a \in H^{|a|}(X)$, $b \in H^{|b|}(Y)$, $c \in H^{|c|}(X)$, and $d \in H^{|d|}(Y)$. This map is extended linearly to be defined on all of $(\tilde{X} \times \tilde{Y}) \times (\tilde{X} \times \tilde{Y})$. To show that m factors through the map q' (to show that we have a well defined map m' that causes the left part of the diagram to commute), we must show that the map m is R-bilinear for fixed third and fourth arguments. That is, let $c \in H^{|c|}(X)$, and $d \in H^{|d|}(Y)$ be fixed. Consider arbitrary $a_1 \in H^{|a_1|}(X), a_2 \in H^{|a_2|}(X)$, and $b \in H^{|b|}(Y)$. If $|a_1| = |a_2|$, then for any $r_1, r_2 \in R$ we have

$$r_1 m((a_1,b),(c,d)) + r_2 m((a_2,b),(c,d)) = r_1(-1)^{|b||c|}(a_1 c \otimes bd) + r_2(-1)^{|b||c|}(a_2 c \otimes bd)$$

$$= (-1)^{|b||c|}(r_1 a_1 c \otimes bd + r_2 a_2 c \otimes bd = (-1)^{|b||c|}(r_1 a_1 + r_2 a_2)c \otimes bd = m((r_1 a_1 + r_2 a_2, b),(c,d)).$$

If $|a_1| \neq |a_2|$, we have that

$$r_1 m((a_1,b),(c,d)) + r_2 m((a_2,b),(c,d)) = m((r_1 a_1 + r_2 a_2, b),(c,d))$$

simply from the fact that m is defined on homogeneous elements and is extended linearly to nonhomogeneous elements. Similarly, given $b_1 \in H^{|b_1|}(X), b_2 \in H^{|b_2|}(X)$, and $a \in H^{|a|}(X)$, we have (for any $r_1, r_2 \in R$)

$$r_1 m((a,b_1),(c,d)) + r_2 m((a,b_2),(c,d)) = m((a, r_1 b_1 + r_2 b_2),(c,d))$$

if either $|b_1| = |b_2|$ or $|b_1| \neq |b_2|$ by the same arguments. This establishes that m is R-bilinear for fixed third and fourth arguments c and d. Thus, we have a well defined map m' such that $m = m' \circ q'$. In the same way, we can show that the map m' induces a well defined map m'' such that $m = m'' \circ q'' \circ q'$. This completes the proof that our multiplication operation on $\tilde{X} \otimes \tilde{Y}$ is well defined.

We will now show that this multiplication operation causes the map $f : \tilde{X} \otimes \tilde{Y} \to \bigoplus_{n \geq 0} H^n(X \times Y)$ to be a ring homomorphism (previously we argued that it is just an R-module homomorphism). To do this, we need only verify that $f((a \otimes b)(c \otimes d)) = f(a \otimes b) \smile f(c \otimes d)$ for simple tensors $a \otimes b$ and $c \otimes d$. The hard part of this verification is that the cup product is supercommutative, which we mentioned before. Here is the computation:

$$\begin{aligned}
f((a \otimes b)(c \otimes b)) &= f((-1)^{|b||c|}(ac \otimes bd) \\
&= (-1)^{|b||c|} f(ac \otimes bd) \\
&= (-1)^{|b||c|}(p_1^*(ac) \smile p_2^*(bd)) \\
&= (-1)^{|b||c|}(p_1^*(a) \smile p_1^*(c) \smile p_2^*(b) \smile p_2^*(d)) \\
&= (-1)^{|b||c|}(p_1^*(a) \smile ((-1)^{|p_1^*(c)||p_2^*(b)|} p_1^*(b) \smile p_2^*(c)) \smile p_2^*(d)) \\
&= (-1)^{|b||c|}(p_1^*(a) \smile ((-1)^{|c||b|} p_1^*(b) \smile p_2^*(c)) \smile p_2^*(d)) \\
&= p_1^*(a) \smile p_1^*(b) \smile p_2^*(c) \smile p_2^*(d) \\
&= f(a \otimes b) \smile f(c \otimes d).
\end{aligned}$$

The fourth to last equality holds because the cup product is supercommutative. The third to last equality holds because $|c| = |p_1^*(c)|$ and $|b| = |p_2^*(b)|$. This establishes that f is a ring homomorphism.

3.3 Cohomology Ring of Product of Spaces

Our goal at this point is to understand the cohomology ring of a product space $X \times Y$ in terms of the cohomology rings of X and Y. To do this, the map f induced by the cross product is the essential tool: $f : (\bigoplus_{n \geq 0} H^n(X)) \otimes (\bigoplus_{n \geq 0} H^n(Y)) \to \bigoplus_{n \geq 0} H^n(X \times Y)$. Where the domain of f has been given the ring structure as described in the previous section, we have that f is a ring homomorphism. We claim under suitable conditions that f is an isomorphism. This will give us a way of computing the cohomology ring of $X \times Y$.

The exact claim we make is as follows: If X and Y are cell complexes and $H^n(Y)$ is a free and finitely generated R-module for all n, then f is a ring isomorphism. We will omit the proof of this because of space. We will, however, show an example of how this claim fails if we drop the finitely generated requirement.

3.4 Importance of $H^n(Y)$ being Finitely Generated

Obviously, it doesn't matter if we require $H^n(X)$ to be free and finitely generated for all n instead of $H^n(Y)$. However, if neither $H^n(X)$ nor $H^n(Y)$ is finitely generated, f need not be an isomorphism in general.

For example, let X be the space consisting of the positive integers with the discrete topology. Let $Y = X$. We have that neither $H^0(X)$ nor $H^0(Y)$ are finitely generated R-modules. We will show that the map f is not surjective. Let ψ be the 0-cocycle defined by $\psi([v]) = 1$ for $v = (i, i)$ with $i \in \mathbb{Z}^+$, and $\phi([v]) = 0$ for all other v. To show that there is no element of $(\bigoplus_{n\geq 0} H^n(X)) \otimes (\bigoplus_{n\geq 0} H^n(Y))$ that f maps to ψ, it suffices to show that no element of $H^0(X) \otimes H^0(Y)$ gets mapped to ψ.

An element of $H^0(X) \otimes H^0(Y)$ can be written in the form $\phi_1 \otimes \phi_1' + ... + \phi_n \otimes \psi_n'$ where $\phi_1, ..., \phi_n \in H^0(X)$ and $\phi_1', ..., \phi_n' \in H^0(Y)$. Since X is discrete, each homology class in $H^0(X)$ consists of just a single function that can be represented by a function from X to R. Because of this, for simplicity we will identify $H^0(X)$ with the set of functions from X to R. The same can be said about $H^0(Y)$ and $H^0(X \times Y)$. Let n be arbitrary. We will show that if $\phi_1, ..., \phi_k$ are functions from X to R and $\phi_1', ..., \phi_k'$ are functions from Y to R, and the function $f(\phi_1 \otimes \phi_1' + ... + \phi_k \otimes \phi_k') = \phi_1\phi_1' + ... + \phi_k\phi_k'$ sends every element of the set $\{(1, 1), ..., (n, n)\}$ to $1 \in R$ and every other element of $X \times Y$ to $0 \in R$, then it must be that $k \geq n$.

Suppose that this is not the case. Let $k < n$ and $\phi_1, ..., \phi_k$, $\phi_1', ..., \phi_k'$ be the appropriate functions. Consider the functions $\phi_1, ..., \phi_k$ restricted to the domain $\{1, ..., n\}$. These functions can be represented by column vectors $v_1, ..., v_k$. Consider the functions $\phi_1', ..., \phi_k'$ restricted to the domain $\{1, ..., n\}$. These functions can be represented by row vectors $v_1', ..., v_k'$. The function $\phi_1\phi_1' + ... + \phi_k\phi_k'$ restricted to the domain $\{(i, j) : 1 \leq i \leq n, 1 \leq j \leq n\}$ can be represented by the $n \times n$ matrix $v_1v_1' + ... + v_kv_k'$. By hypothesis, this is the $n \times n$ identity matrix (over R). Let F be the field of fractions of R. The $n \times n$ identity matrix has rank n over F. Each of the matrices v_iv_i' have rank 1 over F. It is an elementary fact from linear algebra that the sum of k matrices with rank 1 over F has rank at most k over F. Since $k < n$, this is a contradiction. This completes the proof.

3.5 Checking the Cohomology Ring of $S^1 \times S^1$

Previously, we computed the (simplicial) cohomology ring of a triangulation of $S^1 \times S^1$. We can give S^1 a cell complex structure. The cohomology of S^1 as a cell complex is the same as what we computed earlier: $H^0(S^1) = \mathbb{Z}, H^1(S^1) = \mathbb{Z}$. Let a^0 be a generator for $H^0(S^1)$ and let a^1 be a generator for $H^1(S^1)$. We have the following:

$$\begin{aligned} a^0 \smile a^0 &= a^0 \\ a^0 \smile a^1 &= a^1 \\ a^1 \smile a^0 &= a^1 \\ a^1 \smile a^1 &= 0. \end{aligned}$$

Let $A = \bigoplus_{n=0}^{1} H^n(S^1)$. We have that a^0 and a^1 are generators for A. This means that $a^0 \otimes a^0$, $a^0 \otimes a^1$, $a^1 \otimes a^0$, and $a^1 \otimes a^1$ are generators for $A \otimes A$. Let $g, f^1, f^2, h \in A \otimes A$ be defined as follows:

$$\begin{aligned} g &= a^0 \otimes a^0 \\ f^1 &= a^1 \otimes a^0 \\ f^2 &= a^0 \otimes a^1 \\ h &= a^1 \otimes a^1 \end{aligned}$$

Notice that these elements have been named to be reminiscent of the variable names we used when computing the simplicial cohomology of $S^1 \times S^1$. Recall that multiplication of simple tensors in $A \otimes A$ is given as follows: $(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(ac \otimes bd)$. Multiplication in $A \otimes A$ is given as follows:

$$\begin{aligned} &gg = (a^0 \otimes a^0)(a^0 \otimes a^0) = (a^0 \smile a^0) \otimes (a^0 \smile a^0) = a^0 \otimes a^0 = g \\ &gf^1 = (a^0 \otimes a^0)(a^1 \otimes a^0) = (a^0 \smile a^1) \otimes (a^0 \smile a^0) = a^1 \otimes a^0 = f^1 \\ &f^1g = f^1 \\ &gf^2 = f^2 \end{aligned}$$

$$
\begin{aligned}
&f^2 g = g^2\\
&gh = (a^0 \otimes a^0)(a^1 \otimes a^1) = (a^0 \smile a^1) \otimes (a^0 \smile a^1) = a^1 \otimes a^1 = h\\
&hg = h\\
&hf^1 = (a^1 \otimes a^1)(a^1 \otimes a^0) = (-1)(a^1 \smile a^1) \otimes (a^1 \smile a^0) = (-1)0 \otimes a^1 = 0\\
&f^1 h = 0\\
&hf^2 = 0\\
&f^2 h = 0\\
&hh = (a^1 \otimes a^1)(a^1 \otimes a^1) = (-1)(a^1 \smile a^1) \otimes (a^1 \smile a^1) = (-1)0 \otimes 0 = 0.
\end{aligned}
$$

Most interestingly, we have the following:

$$
\begin{aligned}
&f^1 f^2 = (a^1 \otimes a^0)(a^0 \otimes a^1) = (a^1 \smile a^0) \otimes (a^0 \smile a^1) = a^1 \otimes a^1 = h\\
&f^2 f^1 = (a^0 \otimes a^1)(a^1 \otimes a^0) = (-1)(a^0 \smile a^1) \otimes (a^1 \smile a^0) = (-1)a^1 \otimes a^1 = -h.
\end{aligned}
$$

We have now completely described the ring structure of $A \otimes A$. In general, if we know the ring structure of both A and B, then we can compute the ring structure of $A \otimes B$.

Since we are in the situation that each $H^n(S^1)$ is a free and finitely generated $\mathbb{Z}$-module, we have that the ring homomorphism $f : A \otimes A \to \bigoplus_{n\geq 0} H^n(S^1 \times S^1)$ is an isomorphism. Thus, $\tilde{g} = f(g), \tilde{f}^1 = f(f^1), \tilde{f}^2 = f(f^2)$, and $\tilde{h} = f(h)$ generate $\bigoplus_{n\geq 0} H^n(S^1 \times S^1)$. Notice that the multiplication of the elements $\tilde{g}, \tilde{f}^1, \tilde{f}^2$, and $\tilde{h}$ with each other agrees with the multiplication of the simplicial cochains g, f^1, f^2, and h we studied when looking at a triangulation of $S^1 \times S^1$. It should be clear that once this isomorphism between $A \otimes A$ and $\bigoplus_{n\geq 0} H^n(S^1 \times S^1)$ is established, it takes much less work to compute to cohomology ring of $S^1 \times S^1$ using $A \otimes A$ rather than performing simplicial calculations.

4 Notes

This paper follows the development of the cup product as is done in Hatcher's *Algebraic Topology*. The computation of the simplicial cohomology ring of $S^1 \times S^1$ follows the example given in class. Some arguments involving cohomology of simplicial complexes can be found in *Elements of Algebraic Topology* by Munkres. The example of the importance of the cohomology groups being finitely generated when trying to compute the cohomology ring of a product of spaces is an exercise given in Hatcher.

References

[1] Fulton, William. *Algebraic Topology A First Course*. Springer, New York, 1995.

[2] Hatcher, Allen. *Algebraic Topology*. Cambridge Press, Cambridge, United Kingdom, 2002.

[3] Munkres, James R. *Elements of Algebraic Topology*. Perseus Books Publishing, New York, 1984.