CLASSROOM CAPSULES

Using Continuity Induction

Dan Hathaway

We present here a technique we call *continuity induction*, which can be used to prove systematically the fundamental theorems of analysis in the same way ordinary induction is used to prove theorems in discrete mathematics. The method provides a unified way to pass from local properties of the reals to global properties just as ordinary induction passes from local implication (if true for k, the theorem is true for $k + 1$) to a global conclusion. We demonstrate the method by using it to prove the Intermediate Value Theorem and the Heine-Borel Theorem.

Continuity induction has been rediscovered regularly (see the historical notes at the end of this Capsule). With each reappearance, the concept has been refined so that with this version we hope that it may finally be suitable for students in college-level analysis classes.

The method

Principle of continuity induction Let ϕ be a truth-valued function of a real variable, and let $[a, b]$ be a closed interval. If

(1) $\phi(a)$ holds,

(2) for any $x \in [a, b)$, if $\phi(y)$ is true for all $a \leq y \leq x$, then there exists a $\delta > 0$ such that $\phi(y)$ for all $y \in [x, x + \delta)$, and

(3) for any $x \in (a, b]$, if $\phi(y)$ is true for all $a \leq y < x$, then $\phi(x)$,

then $\phi(x)$ is true for all $x \in [a, b]$.

Here is a proof. Suppose (1), (2) and (3) are satisfied. Let $X = \{a \leq y \leq b \mid \phi(y)$ is false$\}$, and let $x = \inf X$. By (1) and (2), $x > a$. For the purpose of contradiction, suppose that $x < b$. By definition of infimum, $\phi(y)$ holds for y in $[a, x)$, so by (3), $\phi(x)$ holds. By (2), for some $\delta > 0$, $\phi(y)$ holds for y in $[a, x + \delta)$, which contradicts the hypothesis that $x = \inf X$. The only remaining possibility is that $x = b$, so $\phi(y)$ holds for all y in $[a, b)$. Then by (3), $\phi(b)$ holds as well, and we are done.

doi:10.4169/college.math.j.42.3.240

The Intermediate Value Theorem Let f be continuous on $[a, b]$ with $f(a) < f(b)$. We will show that if v is such that $f(a) < v < f(b)$, then there exists an x in $[a, b]$ such that $f(x) = v$. On the contrary, however, suppose that $f(x) \neq v$ for all x in $[a, b]$. Let $\phi(x)$ be the statement "$f(x) < v$". Then

(1) $\phi(a)$ holds by hypothesis,

(2) Let $x \in [a, b)$ and suppose $\phi(x)$ holds. Since f is continuous, as is well-known, there must be a $\delta > 0$ such that y in $[x, x + \delta)$ implies $\phi(y)$, and

(3) Let $x \in (a, b]$ and suppose $\phi(y)$ holds for all $y \in [a, x)$. Since f is continuous, it cannot be that $f(x) > v$. By hypothesis, $f(x) \neq v$ so $\phi(x)$ holds.

Thus, $\phi(b)$ holds, which contradicts the assumption $v < f(b)$. ■

The Heine-Borel Theorem Let $A \subset \mathbb{R}$ be a closed and bounded set. Let $\mathcal{O}$ be any open cover of A. Let $a = \inf A$ and $b = \sup A$. Note that a is in A because A is closed. Let $\phi(x)$ be the statement "$[a, x] \cap A$ is covered by a finite subcollection of $\mathcal{O}$."

(1) Since $\mathcal{O}$ covers A and a is in A, there must be some $U \in \mathcal{O}$ such that $a \in U$. Hence, $\{U\} \subseteq \mathcal{O}$ is a finite cover of $\{a\} = [a, a] \cap A$, that is, $\phi(a)$ holds.

(2) Let $x \in [a, b)$ and suppose $\phi(x)$ holds. Because any open cover of $[a, x] \cap A$ also covers $[a, x + \delta) \cap A$ for some δ, there exists $\delta > 0$ such that $\phi(y)$ for all y in $[x, x + \delta)$.

(3) Let $x \in (a, b]$ and suppose $\phi(y)$ holds for all y in $[a, x)$. If $x \notin A$, then there must be some $y < x$ such that $[a, y] \cap A = [a, x] \cap A$ (since A is closed). In this case, $\phi(x)$ holds by assumption. If x is in A, then let $U \in \mathcal{O}$ be such that x is in U and let $B \subseteq U$ be an open interval such that x is in B. Choose $y \in [a, x) \cap B$ and let $C \subseteq \mathcal{O}$ be a finite cover of $[a, y] \cap A$. Then $C \cup \{U\}$ covers $[a, x] \cap A$, so $\phi(x)$ holds.

Thus, $\phi(b)$ holds. That is, $[a, b] \cap A = A$ is covered by a finite subcover of $\mathcal{O}$. ■

Notice that we have the feeling of actually constructing a finite subcover in this proof. That is, an application of step (1) or (3) adds a single open set to a finite subcover, while step (2) does not change the current subcover.

Conclusion Continuity Induction is a general technique that conceptually unifies finding analytic proofs. Generalized induction techniques have this property too. For instance, proofs involving Zorn's Lemma can be reformulated and unified using transfinite induction. Once an induction technique like this is mastered, the object to be constructed in such a proof is obtained in an algorithmic and natural way. As is well known, the difficult point of an inductive-style proof is finding the right induction statement. However, once $\phi(x)$ is chosen correctly, then, as these examples show, (1), (2), and (3) almost prove themselves!

Historical notes

Continuity induction goes back at least to ideas of Lester Ford [**2**]. These were developed by Duren [**1**] and later by Shanahan [**5**], [**6**]. An independent effort, quite close to the present work, is the paper [**4**] by Moss and Roberts. The most recent incarnation is the paper [**3**] by Kalantari, who collected these references and formulated continuity induction using conditions similar to (1) and (2). We encountered Kalantari's work only after completing ours. Our contribution is the addition of condition (3), which makes continuity induction easy to apply to a closed interval.

Summary. Here is a technique for proving the fundamental theorems of analysis that provides a unified way to pass from local properties to global properties on the real line, just as ordinary induction passes from local implication (if true for k, the theorem is true for $k+1$) to a global conclusion in the natural numbers.

References

1. W. L. Duren, Jr., Mathematical induction in sets, *Amer. Math. Monthly* **64** (1957) 19–22. doi:10.2307/2308511
2. L. R. Ford, Interval-Additive propositions, *Amer. Math. Monthly* **64** (1957) 106–108. doi:10.2307/2310392
3. I. Kalantari, Induction over the continuum, in *Induction, Algorithmic Learning Theory, and Philosophy*, M. Friend, N. B. Goethe, and V. S. Harizanov, eds., Springer, Dordrecht Netherlands, 2007, 145–154.
4. R. M. F. Moss and G. T. Roberts, A creeping lemma, *Amer. Math. Monthly* **75** (1968) 649–652. doi:10.2307/2313795
5. P. Shanahan, A unified proof of several basic theorems of real analysis, *Amer. Math. Monthly* **79** (1972) 895–898. doi:10.2307/2317672
6. ———, Addendum to 'A unified proof of several basic theorems of real analysis,' *Amer. Math. Monthly* **81** (1974) 890–891. doi:10.2307/2319452