Interesting Series Divergence Proof

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Background

Here is a relatively simple problem from an analysis class that I managed to prove in a bizzare way. This proof has to deal with an anoying "off-by-one" error.

Theorem

If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} a_n = \infty$, then

$$\sum_{n=1}^{\infty} \frac{a_n}{\sum_{i=1}^{n} a_i} = \infty.$$

Proof

There are two cases: either there exists infinitely many n such that $a_n \geq a_1 + ... + a_{n-1}$ or there does not. If there does, then for each such n we have

$$\frac{a_n}{a_1 + ... + a_n} \geq \frac{1}{2}$$

and so there are infinitely many terms of

$$\sum_{n=1}^{\infty} \frac{a_n}{a_1 + ... + a_n}$$

that are $\geq \frac{1}{2}$ so the sum diverges and hence the theorem holds. Thus, assume that for all but finitely many n that $a_1 + ... + a_{n-1} > a_n$. This gives us the following lemma we need to fix an "off-by-one" issue later:

Lemma: For all but finitely many n, $\frac{a_n}{a_1+...+a_{n-1}} \leq 2\frac{a_n}{a_1+...+a_n}$.
Proof: For each n such that $a_1 + ... + a_{n-1} \geq a_n$, we have $2(a_1 + ... + a_{n-1}) \geq a_1 + ... + a_n$ and so $\frac{1}{a_1+...+a_{n-1}} \leq 2\frac{1}{a_1+...+a_n}$. Since $a_1 + ... + a_{n-1} > a_n$ holds for all but finitely many n, the lemma follows. $\square$

Now for the main part of the proof: Define $f : \mathbb{R}^+ \to \mathbb{R}^+$ by $f(x) = a_{\lfloor x+1 \rfloor}$ ($\lfloor x \rfloor$ is the greatest integer function). We have:

$$\sum_{n=1}^{\infty} a_n = \infty \Rightarrow \int_0^{\infty} f(x)dx = \infty \Rightarrow \lim_{n\to\infty} \int_0^n f(x)dx = \infty$$

$$\Rightarrow \lim_{n\to\infty} \log(\int_0^n f(x)dx) = \infty \Rightarrow \lim_{n\to\infty} \int_1^n [\frac{d}{dy} \log(\int_0^y f(x)dx)]dy = \infty$$

$$\Rightarrow \lim_{n\to\infty} \int_1^n \frac{f(y)}{\int_0^y f(x)dx} dy = \infty$$

We also have the following bound that holds for all but finitely many n (because of the final inequality that uses the above lemma):

$$\int_1^n \frac{f(y)}{\int_0^y f(x)dx}dy \leq \int_1^n \frac{f(y)}{\int_0^{\lfloor y \rfloor} f(x)dx} = \int_1^n \frac{f(y)}{\sum_{i=1}^{\lfloor y \rfloor} a_i}dy$$

$$\leq \int_1^{\lceil n \rceil} \frac{f(y)}{\sum_{i=1}^{\lfloor y \rfloor} a_i}dy = \sum_{j=1}^{\lceil n \rceil -1} \frac{a_{j+1}}{\sum_{i=1}^{j} a_i} = \sum_{k=2}^{\lceil n \rceil -2} \frac{a_k}{\sum_{i=1}^{k-1} a_i} \leq 2 \sum_{k=2}^{\lceil n \rceil -2} \frac{a_k}{\sum_{i=1}^{k} a_i}$$

Since this inequality holds for infinitely many n we have

$$\lim_{n\to\infty} \int_1^n \frac{f(y)}{\int_0^y f(x)dx}dy = \infty \Rightarrow \lim_{n\to\infty} 2 \sum_{k=2}^{\lceil n \rceil -2} \frac{a_k}{\sum_{i=1}^{k} a_i} = \infty \Rightarrow \sum_{n=1}^{\infty} \frac{a_n}{\sum_{i=1}^{n} a_i} = \infty.$$

This completes the theorem. □