Interesting Series Convergence Proof

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Background

Here is a simple problem from an analysis class that I managed to prove in a bizzare way.

Theorem

If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive reals such that $\sum_{n=1}^{\infty} a_n < \infty$, then

$$\sum_{n=1}^{\infty} \frac{a_n}{(\sum_{i=n}^{\infty} a_i)^{\frac{1}{2}}} < \infty.$$

Proof

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $f(x) = a_{\lfloor x \rfloor}$ ($\lfloor x \rfloor$ is the greatest integer function). We have that

$$\sum_{n=1}^{\infty} a_n < \infty \Rightarrow \lim_{n\to\infty} \sum_{j=n}^{\infty} a_j = 0 \Rightarrow \lim_{n\to\infty} \int_n^{\infty} f(x)dx = 0$$

$$\Rightarrow \lim_{n\to\infty} (\int_n^{\infty} f(x)dx)^{\frac{1}{2}} = 0 \Rightarrow \lim_{n\to\infty} \int_n^{\infty} [\frac{d}{dy}(\int_y^{\infty} f(x)dx)^{\frac{1}{2}}]dy = 0$$

$$\Rightarrow \lim_{n\to\infty} \int_n^{\infty} \frac{1}{2} \frac{-f(y)}{(\int_y^{\infty} f(x)dx)^{\frac{1}{2}}} dy = 0 \Rightarrow \lim_{n\to\infty} \int_n^{\infty} \frac{f(y)}{(\int_y^{\infty} f(x)dx)^{\frac{1}{2}}} dy = 0$$

Now, note that:

$$\int_n^{\infty} \frac{f(y)}{(\int_y^{\infty} f(x)dx)^{\frac{1}{2}}} \geq \int_n^{\infty} \frac{f(y)}{(\int_{\lfloor y \rfloor}^{\infty} f(x)dx)^{\frac{1}{2}}} dy = \sum_{j=n}^{\infty} \frac{a_j}{(\sum_{i=j}^{\infty} a_i)^{\frac{1}{2}}}$$

which means

$$\lim_{n\to\infty} \int_n^{\infty} \frac{f(y)}{(\int_y^{\infty} f(x)dx)^{\frac{1}{2}}} dy = 0 \Rightarrow \lim_{n\to\infty} \sum_{j=n}^{\infty} \frac{a_j}{(\sum_{i=j}^{\infty} a_i)^{\frac{1}{2}}} = 0 \Rightarrow \sum_{n=1}^{\infty} \frac{a_n}{(\sum_{i=n}^{\infty} a_i)^{\frac{1}{2}}} < \infty.$$

Putting all the steps together gives us the result. □