Fixed Points of an Integral Transformation With a Triangular Kernel

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March 2010

Background

Here is a relatively simple problem from functional analysis that I managed to prove in an odd way. Perhaps this sort of argument is standard.

Theorem

Let X be the set of all (integrable) functions from $[0,1]$ to $\mathbb{R}$. If $F : X \to X$ is defined by

$$F(f)(x) = \int_0^1 f(t)K(t,x)dt$$

where K is continuous and triangular ($t > x$ implies $K(t,x) = 0$), then F has no non-trivial fixed points.

Proof

Suppose that f is a non-trivial fixed point of F. Since F maps functions to continuous functions, f must be continuous. Let $\tilde{x} = \inf\{x : f(x) \neq 0\}$. Note: $f(\tilde{x}) = 0$. Before proceeding, we need a lemma:

Lemma: Given any $x'' > \tilde{x}$ there is x' s.t. $\tilde{x} < x' \leq x''$ and $|f(x)| \leq |f(x')|$ for all $x \in [\tilde{x}, x']$.
Proof: Just choose x' to be the least x that maximizes $|f(x)|$ in $[\tilde{x}, x'']$, which exists because the interval is compact and f is continuous. Note: $x' > \tilde{x}$ and $|f(x')| > 0$ because of the way $\tilde{x}$ was chosen. □

We are now ready to prove the theorem. Let $M = \max\{|K(t,x)| : t,x \in [0,1]\}$. Let $x'' = \min\{\tilde{x} + \frac{1}{2M}, 1\}$. Apply the above lemma to get x' with the specified properties. We have $F(f)(x') = f(x') \neq 0$ (because $|f(x')| > 0$). On the other hand,

$$|F(f)(x')| = |\int_0^1 f(t)K(t,x')dt| = |\int_{\tilde{x}}^{x'} f(t)K(t,x')dt|$$

$$\leq |x' - \tilde{x}||f(x')|M \leq \frac{1}{2M}|f(x')|M = \frac{1}{2}|f(x')|.$$

This is a contradiction. □